5. Renormalization & Renormalization-group (RG) equations ← hep-ph/0607228

• Renormalization

The factorization formula for thrust τ when $\tau \ll 1$ is given by

$$\frac{d\sigma}{d\tau} = \sigma_0\, H(-s-i0) \int_0^\infty dp_L^2\, J(p_L^2) \int_0^\infty dp_R^2\, J(p_R^2) \int_0^\infty d\omega\, S_T(\omega)\, \delta\left(\tau - \frac{p_L^2+p_R^2+\sqrt{s}\,\omega}{s}\right)$$

Up to NLO, we have

$$H(-s-i0) = |C_V(-s-i0)|^2$$

$$C_V(-s-i0) = 1 + \frac{Z_\alpha \alpha_s}{4\pi} C_F \left[-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-3 - 2\ln\frac{\mu^2}{-s}\right) - \ln^2\frac{\mu^2}{-s} - 3\ln\frac{\mu^2}{-s} + \frac{\pi^2}{6} - 8 + O(\varepsilon)\right]$$

$$J(p^2) = \delta(p^2) + \frac{Z_\alpha \alpha_s}{4\pi} C_F \left\{\frac{4}{\varepsilon^2}\delta(p^2) + \frac{1}{\varepsilon}\left[-4\left(\frac{1}{p^2}\right)_* + 3\,\delta(p^2)\right] + 4\left(\frac{1}{p^2}\ln\frac{p^2}{\mu^2}\right)_* - 3\left(\frac{1}{p^2}\right)_* + (7-\pi^2)\,\delta(p^2)\right\}$$

$$S_T(\omega) = \delta(\omega) + \frac{Z_\alpha \alpha_s}{4\pi} C_F \frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)} \frac{8}{\varepsilon} \frac{1}{\omega}\left(\frac{\omega}{\mu}\right)^{-2\varepsilon}$$

$$S_T(\omega) = \delta(\omega) + \frac{Z_\alpha \alpha_s}{4\pi} C_F \left[-\frac{4}{\varepsilon^2}\delta(\omega) + \frac{8}{\varepsilon}\left(\frac{1}{\omega}\right)_* - 16\left(\frac{1}{\omega}\ln\frac{\omega}{\mu}\right)_* + \frac{\pi^2}{3}\delta(\omega)\right]$$

We have used the renormalization of $\alpha_s \equiv \alpha_s(\mu)$

$$\alpha_{s,0} = Z_\alpha\, \alpha_s(\mu)\, \mu^{2\varepsilon}, \qquad \text{with} \quad Z_\alpha = 1 + \frac{\alpha_s}{4\pi}\left(-\frac{\beta_0}{\varepsilon}\right) + \cdots$$

The physical observables is free of divergences, so the cross section can be expressed by renormalized quantities:

$$\frac{d\sigma}{d\tau} = \sigma_0\, |Z_V(\mu)\, C_V(-s-i0,\mu)|^2 \int_0^\infty dp_L^2 \int_0^\infty dp_R^2 \int_0^\infty d\omega\, \delta\left(\tau - \frac{p_L^2+p_R^2+\sqrt{s}\,\omega}{s}\right) \int_0^\infty d\omega'\, Z_S^{-1}(\omega,\omega',\mu)\, S_T(\omega',\mu)$$

$$\times \int_0^\infty dp_L'^2\, Z_J^{-1}(p_L^2, p_L'^2,\mu)\, J(p_L'^2,\mu) \int_0^\infty dp_R'^2\, Z_J^{-1}(p_R^2, p_R'^2,\mu)\, J(p_R'^2,\mu)$$

$$= \sigma_0\, |C_V(-s-i0,\mu)|^2 \int_0^\infty dp_L'^2 \int_0^\infty dp_R'^2 \int_0^\infty d\omega'\, \delta\left(\tau - \frac{p_L'^2+p_R'^2+\sqrt{s}\,\omega'}{s}\right) J(p_L'^2,\mu)\, J(p_R'^2,\mu)\, S_T(\omega',\mu)$$

$$\Rightarrow |Z_V(\mu)|^2 \int_0^\infty dp_L^2 \int_0^\infty dp_R^2 \int_0^\infty d\omega\, \delta\left(\tau - \frac{p_L^2+p_R^2+\sqrt{s}\,\omega}{s}\right) Z_S^{-1}(\omega,\omega',\mu)\, Z_J^{-1}(p_L^2,p_L'^2,\mu)\, Z_J^{-1}(p_R^2,p_R'^2,\mu) = \delta\left(\tau - \frac{p_L'^2+p_R'^2+\sqrt{s}\,\omega'}{s}\right)$$

• RGE of hard function

The renormalization of hard function is local. In $\overline{MS}$ scheme, the poles are absorbed into $Z_V(\mu)$, leading to

$$Z_V(\mu) = 1 + \frac{C_F \alpha_s}{4\pi}\left[-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-3 - 2\ln\frac{\mu^2}{-s}\right)\right] + O(\alpha_s^2)$$

To derive the RGE of hard function, we first notice that the bare hard function is RG invariant

$$\frac{d}{d\ln\mu} C_V(-s-i0) = 0 \quad \Rightarrow \quad \frac{d}{d\ln\mu} \ln\left[Z_V(\mu)\, C_V(-s-i0,\mu)\right] = 0$$

$$\Rightarrow \quad \frac{d}{d\ln\mu} \ln C_V(-s-i0,\mu) = -Z_V^{-1}(\mu) \frac{d}{d\ln\mu} Z_V(\mu)$$

$$\Rightarrow \quad \frac{d}{d\ln\mu} C_V(-s-i0,\mu) = \Gamma_V(\mu) \cdot C_V(-s-i0,\mu)$$

$\Gamma_V(\mu)$ can be obtained from $Z_V(\mu)$

$$\Gamma_V(\mu) = -Z_V^{-1}(\mu) \frac{d}{d\ln\mu} Z_V(\mu)$$

$$= -\left\{1 - \frac{C_F\alpha_s}{4\pi}\left[-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-3 - 2\ln\frac{\mu^2}{-s}\right)\right]\right\} \frac{d}{d\ln\mu}\left\{1 + \frac{C_F\alpha_s}{4\pi}\left[-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-3 - 2\ln\frac{\mu^2}{-s}\right)\right]\right\}$$

$$= -\frac{d\alpha_s}{d\ln\mu} \cdot \frac{C_F}{4\pi}\left[-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-3 - 2\ln\frac{\mu^2}{-s}\right)\right] - \frac{C_F\alpha_s}{4\pi}\left(-\frac{4}{\varepsilon}\right)$$

$\frac{d\alpha_s}{d\ln\mu} = \beta(\alpha_s) - 2\varepsilon\alpha_s$

$\beta(\alpha_s) = -2\alpha_s \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}$

$$= -\frac{C_F\alpha_s}{4\pi}\left\{\left(-2\varepsilon + O(\alpha_s)\right)\left[-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(\underline{-3 - 2\ln\frac{\mu^2}{-s}}\right)\right] - \frac{4}{\varepsilon}\right\}$$

$\Gamma_V = 2\alpha_s \frac{\partial Z_V^{[1]}(\alpha_s)}{\partial \alpha_s}$

$$= \frac{\alpha_s}{4\pi}\left(4C_F \ln\frac{-s}{\mu^2} - 6C_F\right)$$

$$\Rightarrow \quad \Gamma_V(\mu) = \Gamma_{cusp}(\alpha_s) \ln\frac{-s}{\mu^2} + \gamma^V(\alpha_s)$$

The higher-order results of Γ_{cusp} and γ^V have already been obtained

$$\Gamma_{cusp}(\alpha_s) = \sum_{n=0}^{\infty} \Gamma_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}, \qquad \gamma^V(\alpha_s) = \sum_{n=0}^{\infty} \gamma_n^V \left(\frac{\alpha_s}{4\pi}\right)^{n+1}$$

$\Gamma_0 = 4C_F$ $\qquad$ $\gamma_0^V = -6C_F$

$\Gamma_1, \Gamma_2, \Gamma_3$ are known $\qquad$ $\gamma_1^V, \gamma_2^V, \gamma_3^V$ are known.

The solution of RGE is

$$C_V(-s-i0,\mu) = C_V(-s-i0,\mu_h)\, e^{\int_{\mu_h}^{\mu} \frac{d\mu'}{\mu'} \Gamma_V(\mu')}$$

$$\int_{\nu}^{\mu} \frac{d\mu'}{\mu'} \Gamma_V(\mu') = \int_{\nu}^{\mu} \frac{d\mu'}{\mu'} \left[\Gamma_{cusp}(\alpha_s(\mu')) \ln\frac{-s}{\mu'^2} + \gamma^V(\alpha_s(\mu'))\right]$$

$$= \int_{\alpha_s(\nu)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)}\left[\Gamma_{cusp}(\alpha) \ln\frac{-s}{\mu'^2} + \gamma^V(\alpha)\right]$$

$$= -2\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{cusp}(\alpha)}{\beta(\alpha)} \int_{\alpha_s(-s)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} + \int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\gamma^V(\alpha)}{\beta(\alpha)}$$

$\frac{d\alpha_s}{d\ln\mu} = \beta(\alpha_s)$

$d\ln\mu = \frac{d\alpha_s}{\beta(\alpha_s)} \Rightarrow \ln\frac{\mu'^2}{-s} = 2\int_{\alpha_s(-s)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}$ with $\alpha = \alpha_s(\mu')$

For convenience, we define

$$S_\Gamma(\nu,\mu) = -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{cusp}(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\nu)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}, \qquad a_{\gamma^V}(\nu,\mu) = -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\gamma^V(\alpha)}{\beta(\alpha)}$$

Then we have

$$\int_{\nu}^{\mu} \frac{d\mu'}{\mu'} \Gamma_V(\mu') = -2\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\Gamma_{cusp}(\alpha)}{\beta(\alpha)}\left[\int_{\alpha_s(\nu)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')} - \int_{\alpha_s(\nu)}^{\alpha_s(-s)} \frac{d\alpha'}{\beta(\alpha')}\right] - a_{\gamma^V}(\nu,\mu)$$

$$= 2S_\Gamma(\nu,\mu) - a_\Gamma(\nu,\mu) \ln\frac{-s}{\nu^2} - a_{\gamma^V}(\nu,\mu)$$

Finally, the solution of RGE for hard coefficient:

$$C_V(-s-i0,\mu) = C_V(-s-i0,\mu_h)\exp\left[2S_\Gamma(\mu_h,\mu) - a_{\gamma^V}(\mu_h,\mu) - a_\Gamma(\mu_h,\mu)\ln\frac{-s}{\mu_h^2}\right]$$

$$\Rightarrow C_V(-s-i0,\mu) = \exp\left[2S_\Gamma(\mu_h,\mu) - a_{\gamma^V}(\mu_h,\mu)\right]\left(\frac{-s}{\mu_h^2}\right)^{-a_\Gamma(\mu_h,\mu)} \underline{C_V(-s-i0,\mu_h)}$$

$\mu_h^2 \sim s$ to eliminate large logs

The exponents $S_\Gamma(\nu,\mu)$ and $a_\gamma(\nu,\mu)$ can be obtained by expanding the anomalous dimensions in α_s and performing integrals order by order

$$\frac{1}{\beta(\alpha)} = \frac{4\pi}{-2\alpha^2\beta_0}\frac{1}{1+\frac{\alpha}{4\pi}\frac{\beta_1}{\beta_0}+\left(\frac{\alpha}{4\pi}\right)^2\frac{\beta_2}{\beta_0}+\cdots} = -\frac{2\pi}{\beta_0}\frac{1}{\alpha^2}\left(1-\frac{\alpha}{4\pi}\frac{\beta_1}{\beta_0}+\mathcal{O}(\alpha^2)\right)$$

$$\begin{aligned}
S_\Gamma(\nu,\mu) &= -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{cusp}(\alpha)}{\beta(\alpha)}\int_{\alpha_s(\nu)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')} \\
&= -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\left(-\frac{1}{2\beta_0}\right)\frac{1}{\alpha}\left(1-\frac{\alpha}{4\pi}\frac{\beta_1}{\beta_0}+\mathcal{O}(\alpha^2)\right)\left(\Gamma_0+\frac{\alpha}{4\pi}\Gamma_1+\mathcal{O}(\alpha^2)\right) \\
&\quad\times\underline{\int_{\alpha_s(\nu)}^{\alpha} d\alpha'\left(-\frac{2\pi}{\beta_0}\right)\frac{1}{\alpha'^2}\left(1-\frac{\alpha}{4\pi}\frac{\beta_1}{\beta_0}+\mathcal{O}(\alpha^2)\right)} = -\frac{2\pi}{\beta_0}\left[\frac{1}{\alpha_s(\nu)}-\frac{1}{\alpha}-\frac{1}{4\pi}\frac{\beta_1}{\beta_0}\ln\frac{\alpha}{\alpha_s(\nu)}+\mathcal{O}(\alpha)\right] \\
&= \frac{\pi}{\beta_0^2}\Gamma_0\left[\frac{1}{\alpha_s(\nu)}-\frac{1}{\alpha_s(\mu)}-\frac{1}{\alpha_s(\nu)}\ln\frac{\alpha_s(\mu)}{\alpha_s(\nu)}\right]+\cdots
\end{aligned}$$

$$\Rightarrow S_\Gamma(\nu,\mu) = \frac{\Gamma_0}{4\beta_0^2}\left\{\frac{4\pi}{\alpha_s(\nu)}\left[1-\frac{\alpha_s(\nu)}{\alpha_s(\mu)}-\ln\frac{\alpha_s(\mu)}{\alpha_s(\nu)}\right]+\cdots\right.$$

← higher-order result see hep-ph/0607228

• RGE of jet function

hep/ph/0603140 hep-ph/0607228

Renomalization:

$$\begin{aligned}
J(p^2,\mu) &= \int_0^\infty dp'^2\, Z_J(p^2,p'^2,\mu)\, J(p'^2) \\
&= Z_J(p^2,p'^2,\mu)\otimes J(p'^2)
\end{aligned}$$

with $Z_J(p^2,p'^2,\mu) = \delta(p^2-p'^2) + \frac{C_F\alpha_s}{4\pi}\left\{-\frac{4}{\varepsilon^2}\delta(p^2-p'^2)+\frac{1}{\varepsilon}\left[4\left(\frac{1}{p^2-p'^2}\right)_*-3\,\delta(p^2-p'^2)\right]\right\}$

$$\begin{aligned}
\Rightarrow \frac{d}{d\ln\mu}J(p^2,\mu) &= \left[\frac{d}{d\ln\mu}Z_J(p^2,p'^2,\mu)\right]\otimes J(p'^2) \\
&= \left[\frac{d}{d\ln\mu}Z_J(p^2,p''^2,\mu)\right]\otimes Z_J(p''^2,p'^2,\mu)\otimes J(p'^2,\mu)
\end{aligned}$$

$$\Rightarrow \frac{d}{d\ln\mu}J(p^2,\mu) = -\Gamma_J(p^2,p'^2,\mu)\otimes J(p'^2,\mu)$$

$$\begin{aligned}
\text{with}\quad \Gamma_J(p^2,p'^2,\mu) &= -\left[\frac{d}{d\ln\mu}Z_J(p^2,p''^2,\mu)\right]\otimes Z_J(p''^2,p'^2,\mu) \\
&= 2\alpha_s\frac{\partial}{\partial\alpha_s}Z_J^{[1]}(p^2,p'^2,\mu) \\
&= \frac{C_F\alpha_s}{4\pi}\left[8\left(\frac{1}{p^2-p'^2}\right)_*-6\,\delta(p^2-p'^2)\right]
\end{aligned}$$

$$\Rightarrow \Gamma_J(p^2,p'^2,\mu) = 2\,\Gamma_{cusp}(\alpha_s)\left(\frac{1}{p^2-p'^2}\right)_* + 2\gamma^J(\alpha_s)\,\delta(p^2-p'^2)$$

with $\gamma^J(\alpha_s) = \sum_{n=0}^{\infty}\gamma_n^J\left(\frac{\alpha_s}{4\pi}\right)^{n+1}$, $\gamma_0^J = -3C_F$

Then the RGE of jet function can be written explicitly:

$$\frac{d}{d\ln\mu} J(p^2,\mu) = -\int_0^\infty dp'^2 \left[2\Gamma_{cusp}(\alpha_s) \left(\frac{1}{p^2-p'^2}\right)_* + 2\gamma^J(\alpha_s)\,\delta(p^2-p'^2) \right] J(p'^2,\mu)$$

To make the convolution of star function clear, we change variable from p'^2 to $p''^2 = p^2 - p'^2$

$$\begin{aligned}
\int_0^\infty dp'^2 \left(\frac{1}{p^2-p'^2}\right)_* J(p^2) &= \int_{-\infty}^{p^2} dp''^2 \left(\frac{1}{p''^2}\right)_* J(p^2-p''^2,\mu) \\
&= \int_0^{p^2} dp''^2 \left(\frac{1}{p''^2}\right)_* J(p^2-p''^2,\mu) \\
&= \int_0^{p^2} dp''^2 \frac{J(p^2-p''^2,\mu) - J(p^2,\mu)}{p''^2} - J(p^2,\mu) \int_{p^2}^{\mu^2} dp''^2 \frac{1}{p''^2} \\
&= \int_0^{p^2} dp'^2 \frac{J(p'^2,\mu) - J(p^2,\mu)}{p^2-p'^2} + J(p^2,\mu) \ln\frac{p^2}{\mu^2}
\end{aligned}$$

$$\Rightarrow \frac{d}{d\ln\mu} J(p^2,\mu) = -\left[2\Gamma_{cusp}(\alpha_s) \ln\frac{p^2}{\mu^2} - 2\gamma^J(\alpha_s) \right] J(p^2,\mu) - 2\Gamma_{cusp}(\alpha_s) \int_0^{p^2} dp'^2 \frac{J(p'^2,\mu) - J(p^2,\mu)}{p^2-p'^2}$$

To solve this integro-differential equation. Laplace transformation is helpful

$$\tilde{j}\left(\ln\frac{1}{e^{\gamma_E} s\mu^2},\mu\right) = \int_0^\infty dp^2\, e^{-sp^2} J(p^2,\mu) \quad .$$

the inverse transformation is

$$J(p^2,\mu) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} ds\, e^{sp^2}\, \tilde{j}\left(\ln\frac{1}{e^{\gamma_E} s\mu^2},\mu\right) \qquad \text{with } C>0$$

To derive the jet function in Laplace space. we can compare the following two expansion:

$$\begin{aligned}
\int_0^\infty dp^2\, e^{-sp^2} \frac{1}{p^2}\left(\frac{p^2}{\mu^2}\right)^{-\varepsilon} &= \int_0^\infty dp^2\, e^{-sp^2} \left[-\frac{1}{\varepsilon}\delta(p^2) + \left(\frac{1}{p^2}\right)_* - \varepsilon \left(\frac{1}{p^2}\ln\frac{p^2}{\mu^2}\right)_* + \cdots \right] \\
&= (e^{\gamma_E} s\mu^2)^{\varepsilon} e^{-\varepsilon\gamma_E}\,\Gamma(-\varepsilon) = \left[-\frac{1}{\varepsilon} + \ln\frac{1}{e^{\gamma_E} s\mu^2} - \varepsilon\left(\frac{\pi^2}{12} + \frac{1}{2}\ln^2\frac{1}{e^{\gamma_E} s\mu^2} \right) + \cdots \right]
\end{aligned}$$

So the jet function in Laplace space can be obtained from the on in momentum space by following replacement:

$$\delta(p^2) \to 1, \qquad \left(\frac{1}{p^2}\right)_* \to L, \qquad \left(\frac{1}{p^2}\ln\frac{p^2}{\mu^2}\right)_* \to \frac{1}{2}L^2 + \frac{\pi^2}{12}, \qquad \text{with } L = \ln\frac{1}{e^{\gamma_E} s\mu^2}$$

$$\tilde{j}(L) = 1 + \frac{\alpha_s}{4\pi} C_F \left[\frac{4}{\varepsilon^2} + \frac{1}{\varepsilon}(-4L+3) + 2L^2 - 3L + 7 - \frac{2\pi^2}{3} \right] + O(\alpha_s^2)$$

There is no star function in Laplace space, so the renormalization is local

$$\tilde{j}(L,\mu) = Z_j(\mu)\, \tilde{j}(L)$$

with $Z_j(\mu) = 1 + \frac{\alpha_s}{4\pi} C_F \left[-\frac{4}{\varepsilon^2} + \frac{1}{\varepsilon}(4L-3) \right] + O(\alpha_s^2)$

The corresponding RGE is

$$\frac{d}{d\ln\mu} \tilde{j}(L,\mu) = -\Gamma_j(\mu)\, \tilde{j}(L,\mu)$$

with $\Gamma_j(\mu) = 2\alpha_s \frac{\partial Z_j^{[1]}}{\partial \alpha_s} = 2\,\Gamma_{cusp}(\alpha_s)\,L + 2\gamma^J(\alpha_s)$

The solution of jet function is similar with the one of hard coefficient.

$$\tilde{j}\left(\ln\frac{1}{e^{\gamma_E} s\mu^2},\mu\right) = \exp\left[-4S_\Gamma(\mu_j,\mu) + 2a_{\gamma^J}(\mu_j,\mu)\right]\left(\frac{1}{e^{\gamma_E} s\mu_j^2}\right)^{2a_\Gamma(\mu_j,\mu)}\tilde{j}\left(\ln\frac{1}{e^{\gamma_E} s\mu_j^2},\mu_j\right)$$

$$= \exp\left[-4S_\Gamma(\mu_j,\mu) + 2a_{\gamma^J}(\mu_j,\mu)\right]\tilde{j}(\partial_{\eta_j},\mu_j)\left(\frac{1}{e^{\gamma_E} s\mu_j^2}\right)^{\eta_j}$$

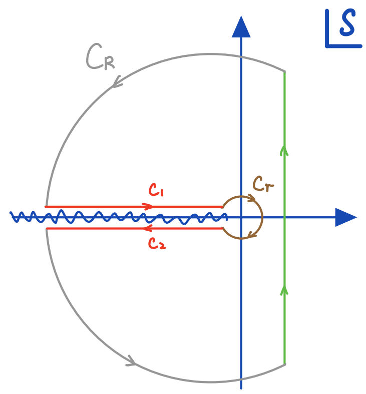
with $\eta_j = 2a_\Gamma(\mu_j,\mu)$

To obtain the evolved jet function in momentum space, we perform inverse Laplace transformation:

$$J(p^2,\mu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, e^{sp^2}\,\tilde{j}\left(\ln\frac{1}{e^{\gamma_E} s\mu^2},\mu\right)$$

$$= \exp\left[-4S_\Gamma(\mu_j,\mu) + 2a_{\gamma^J}(\mu_j,\mu)\right]\tilde{j}(\partial_{\eta_j},\mu_j)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, e^{sp^2}\left(\frac{1}{e^{\gamma_E} s\mu_j^2}\right)^{\eta_j}$$

Thus, we focus on $I(p^2,\eta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, e^{sp^2}\, s^{-\eta}$



$$\left(\int_{c-i\infty}^{c+i\infty} ds + \int_{C_R} ds + \int_{C_r} ds + \int_{C_1} ds + \int_{C_2} ds\right) e^{sp^2}\, s^{-\eta} = 0$$

$\int_{C_r} ds$ gives zero contribution due to $|\eta|<1$

$$\Rightarrow\ I(p^2,\eta) = -\frac{1}{2\pi i}\left(\int_{C_1} ds + \int_{C_2} ds\right) e^{sp^2}\, s^{-\eta}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{0} ds\, e^{sp^2}\left[-(s+i0)^{-\eta} + (s-i0)^{-\eta}\right]$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{0} ds\, e^{sp^2}\,(-s)^{-\eta}\, 2i\sin(\pi\eta)$$

$$= \frac{\sin\pi\eta}{\pi}\int_0^{\infty} dt\, e^{-tp^2}\, t^{-\eta}$$

$$= \frac{\sin\pi\eta}{\pi}\,\Gamma(1-\eta)\,(p^2)^{-1+\eta}$$

$$\Rightarrow\ I(p^2,\eta) = (p^2)^{-1+\eta}\,\frac{1}{\Gamma(\eta)}$$

$(s+i0)^{-\eta} - (s-i0)^{-\eta}$

$= e^{-\eta\ln(s+i0)} - e^{-\eta\ln(s-i0)}$

$= e^{-\eta[\ln(-s)+i\pi]} - e^{-\eta[\ln(-s)-i\pi]}$

$= (-s)^{-\eta}\left(e^{-i\pi\eta} - e^{i\pi\eta}\right)$

$= (-s)^{-\eta}\left(-2i\sin\pi\eta\right)$

Finally, the evolved jet function is given by

$$J(p^2,\mu) = \exp\left[-4S_\Gamma(\mu_j,\mu) + 2a_{\gamma^J}(\mu_j,\mu)\right]\tilde{j}(\partial_{\eta_j},\mu_j)\,\frac{1}{p^2}\left(\frac{p^2}{\mu_j^2}\right)^{\eta_j}\frac{e^{-\gamma_E\eta_j}}{\Gamma(\eta_j)}$$

- <u>RGE of soft function</u>

The soft function in Laplace space is given by

$$\tilde{S}_T\left(\ln\frac{1}{e^{\gamma_E} s\mu}, \mu\right) = \int_0^\infty d\omega\, e^{-s\omega} S_T(\omega,\mu) \quad .$$

$$\Rightarrow \quad \tilde{S}_T(L) = 1 + \frac{C_F\alpha_s}{4\pi}\left(-\frac{4}{\varepsilon^2} + \frac{8}{\varepsilon}L - 8L^2 - \pi^2\right)$$

$$\tilde{S}_T(L,\mu) = Z_{\tilde{S}}(\mu)\,\tilde{S}_T(L) \quad , \text{ with } L = \ln\frac{1}{e^{\gamma_E} s\mu}$$

Here we get the replacement rule from the following expansion

$$\int_0^\infty d\omega\, e^{-s\omega}\frac{1}{\omega}\left(\frac{\omega}{\mu}\right)^{-2\varepsilon} = \int_0^\infty d\omega\, e^{-s\omega}\left[-\frac{1}{2\varepsilon}\delta(\omega) + \left(\frac{1}{\omega}\right)_* - 2\varepsilon\left(\frac{1}{\omega}\ln\frac{\omega}{\mu}\right)_* + \cdots\right]$$

$$= (e^{\gamma_E} s\mu)^{2\varepsilon} e^{-2\varepsilon\gamma_E}\,\Gamma(-2\varepsilon) = \left[-\frac{1}{2\varepsilon} + \ln\frac{1}{e^{\gamma_E} s\mu} - 2\varepsilon\left(\frac{\pi^2}{12} + \frac{1}{2}\ln^2\frac{1}{e^{\gamma_E} s\mu}\right) + \cdots\right]$$

Renormalization factor is:

$$Z_{\tilde{S}}(\mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left(\frac{4}{\varepsilon^2} - \frac{8}{\varepsilon}L\right) + \mathcal{O}(\alpha_s^2)$$

RGE in Laplace space is

$$\frac{d}{d\ln\mu}\tilde{S}_T(L,\mu) = -\Gamma_{\tilde{S}}(\mu)\,\tilde{S}_T(L,\mu)$$

Anomolous dimension is

$$\Gamma_{\tilde{S}}(\mu) = 2\alpha_s\frac{\partial Z_{\tilde{S}}^{[1]}}{\partial\alpha_s} = \frac{\alpha_s}{4\pi}C_F(-16L)$$

$$= -4\,\Gamma_{cusp}(\alpha_s) + 2\gamma^S(\alpha_s) \qquad \text{with } \gamma^S(\alpha_s) = \sum_{n=0}^\infty \gamma_n^S\left(\frac{\alpha_s}{4\pi}\right)^{n+1}, \quad \gamma_0^S = 0$$

The solution of RGE in Laplace space is

$$\tilde{S}_T\left(\ln\frac{1}{e^{\gamma_E} s\mu}, \mu\right) = \exp\left[4S_\Gamma(\mu_s,\mu) + 2a_{\gamma^S}(\mu_s,\mu)\right]\left(\frac{1}{e^{\gamma_E} s\mu_s}\right)^{-4a_\Gamma(\mu_j,\mu)}\tilde{S}_T\left(\ln\frac{1}{e^{\gamma_E} s\mu_s}, \mu_s\right)$$

$$= \exp\left[4S_\Gamma(\mu_s,\mu) + 2a_{\gamma^S}(\mu_s,\mu)\right]\tilde{S}_T(\partial_{\eta_s},\mu_s)\left(\frac{1}{e^{\gamma_E} s\mu_s}\right)^{\eta_s}$$

with $\eta_s = -4a_\Gamma(\mu_s,\mu)$

After inverse transformation, we get the evolved soft function in momentum space:

$$S_T(\omega,\mu) = \exp\left[4S_\Gamma(\mu_s,\mu) + 2a_{\gamma^S}(\mu_s,\mu)\right]\tilde{S}_T(\partial_{\eta_s},\mu_s)\,\frac{1}{\omega}\left(\frac{\omega}{\mu_s}\right)^{\eta_s}\frac{e^{-\gamma_E\eta_s}}{\Gamma(\eta_s)}$$

- **Resummation for thrust observable:** See hep-ph: 0803.0342, Bech, Schwartz

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = |C_V(-s-i0)|^2 \int_0^\infty dP_L^2 \int_0^\infty dP_R^2 \int_0^\infty dw\, \delta\!\left(\tau - \frac{P_L^2+P_R^2+\sqrt{s}\,w}{s}\right) J(P_L^2,\mu)\, J(P_R^2,\mu)\, S_T(\omega,\mu)$$

$$= \exp\left[4S_\Gamma(\mu_h,\mu) - 2a_{\gamma^V}(\mu_h,\mu) - 8S_\Gamma(\mu_j,\mu) + 4a_{\gamma^J}(\mu_j,\mu) + 4S_\Gamma(\mu_s,\mu) + 2a_{\gamma^s}(\mu_s,\mu)\right]$$

$$\times \left(\frac{s}{\mu_h^2}\right)^{-2a_\Gamma(\mu_h,\mu)} H(s)\, \tilde{j}(\partial_{\eta_{j_1}},\mu_j)\, \tilde{j}(\partial_{\eta_{j_2}},\mu_j)\, \tilde{S}_T(\partial_{\eta_s},\mu_s)\, \frac{e^{-\gamma_E(\eta_{j_1}+\eta_{j_2}+\eta_s)}}{\Gamma(\eta_{j_1})\Gamma(\eta_{j_2})\Gamma(\eta_s)} (\mu_j^2)^{-\eta_{j_1}-\eta_{j_2}}\, \mu_s^{-\eta_s}$$

$$\times \sqrt{s}\,(\sqrt{s})^{1-\eta_s} \underbrace{\int_0^{\tau s} dP_L^2\, (P_L^2)^{-1+\eta_{j_1}} \underbrace{\int_0^{\tau s - P_L^2} dP_R^2\, (P_R^2)^{-1+\eta_{j_2}} (\tau s - P_L^2 - P_R^2)^{-1+\eta_s}}_{(\tau s - P_L^2)^{-1+\eta_{j_2}+\eta_s}\frac{\Gamma(\eta_{j_2})\Gamma(\eta_s)}{\Gamma(\eta_{j_2}+\eta_s)}}}_{(\tau s)^{-1+\eta_{j_1}+\eta_{j_2}+\eta_s}\frac{\Gamma(\eta_{j_1})\Gamma(\eta_{j_2})\Gamma(\eta_s)}{\Gamma(\eta_{j_1}+\eta_{j_2}+\eta_s)}}$$

$$= \exp\left[4S_\Gamma(\mu_h,\mu_j) + 4S_\Gamma(\mu_s,\mu_j) + 4a_{\gamma^J}(\mu_j,\mu_h) + 2a_{\gamma^s}(\mu_s,\mu_h)\right] \left(\frac{\mu_j^2}{\mu_h\mu_s}\right)^{4a_\Gamma(\mu_j,\mu)} \left(\frac{s}{\mu_h^2}\right)^{-2a_\Gamma(\mu_h,\mu)}$$

$$\times H(s)\, \tilde{j}(\partial_{\eta_{j_1}},\mu_{j_1})\, \tilde{j}(\partial_{\eta_{j_2}},\mu_j)\, \tilde{S}_T(\partial_{\eta_s},\mu_s)\, \underline{(\sqrt{s})^{-\eta_s} (\mu_j^2)^{-\eta_{j_1}-\eta_{j_2}}\, \mu_s^{-\eta_s}\, \frac{1}{\tau}(\tau s)^{\eta}}\, \frac{e^{-\gamma_E\eta}}{\Gamma(\eta)}$$

$$= (\sqrt{s})^{\eta_{j_1}+\eta_{j_2}} (\mu_j^2)^{-\eta_{j_1}-\eta_{j_2}}\, \mu_s^{\eta_{j_1}+\eta_{j_2}}\, \frac{1}{\tau}\left(\frac{\tau\sqrt{s}}{\mu_s}\right)^{\eta}$$

$$= \left(\frac{\mu_s\sqrt{s}}{\mu_j^2}\right)^{\eta_{j_1}+\eta_{j_2}} \frac{1}{\tau}\left(\frac{\tau\sqrt{s}}{\mu_s}\right)^{\eta} \qquad \eta_j = 2a_\Gamma(\mu_j,\mu)$$

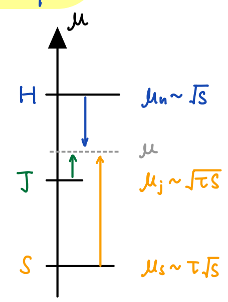
$$\left(\frac{\mu_s\sqrt{s}}{\mu_j^2}\right)^{4a_\Gamma(\mu_j,\mu)} \left(\frac{\mu_j^2}{\mu_h\mu_s}\right)^{4a_\Gamma(\mu_j,\mu)} \left(\frac{s}{\mu_h^2}\right)^{-2a_\Gamma(\mu_h,\mu)} = \left(\frac{s}{\mu_h^2}\right)^{2a_\Gamma(\mu_j,\mu_h)}$$

$$\Rightarrow \frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \exp\left[4S_\Gamma(\mu_h,\mu_j) + 4S_\Gamma(\mu_s,\mu_j) + 4a_{\gamma^J}(\mu_j,\mu_h) + 2a_{\gamma^s}(\mu_s,\mu_h)\right] \left(\frac{s}{\mu_h^2}\right)^{2a_\Gamma(\mu_j,\mu_h)}$$

$$\times H(s)\, \tilde{j}\!\left(\ln\frac{\mu_s\sqrt{s}}{\mu_j^2}+\partial_\eta,\mu_{j_1}\right) \tilde{j}\!\left(\ln\frac{\mu_s\sqrt{s}}{\mu_j^2}+\partial_\eta,\mu_j\right) \tilde{S}_T(\partial_\eta,\mu_s)\, \frac{1}{\tau}\left(\frac{\tau\sqrt{s}}{\mu_s}\right)^{\eta} \frac{e^{-\gamma_E\eta}}{\Gamma(\eta)}$$

μ-dependence cancels out!

with $\eta = \eta_{j_1} + \eta_{j_2} + \eta_s = 4a_\Gamma(\mu_j,\mu_s)$



Scale choice: $\mu_h \sim \sqrt{s}$, $\mu_s \sim \tau\sqrt{s}$, $\mu_j \sim \sqrt{\mu_s\sqrt{s}} = \sqrt{\tau}\sqrt{s}$

where we have used

$$S_\Gamma(\mu_1,\mu) - S_\Gamma(\mu_2,\mu) = S_\Gamma(\mu_1,\mu_2) + \ln\frac{\mu_2}{\mu_1}\, a_\Gamma(\mu_2,\mu)$$

$$2\gamma^V - 4\gamma^J - 2\gamma^s = 0 \quad \Rightarrow \quad 2a_{\gamma^V}(\mu_h,\mu) = 4a_{\gamma^J}(\mu_h,\mu) + 2a_{\gamma^s}(\mu_s,\mu)$$

$$-2a_{\gamma^V}(\mu_h,\mu) + 4a_{\gamma^J}(\mu_j,\mu) + 2a_{\gamma^s}(\mu_s,\mu) = 4a_{\gamma^J}(\mu_j,\mu_h) + 2a_{\gamma^s}(\mu_s,\mu_h)$$

- Compare with conventional approach using Mellin transformation

$$\tilde{\sigma}_N(\mu) = \int_0^\infty d\tau\, \tau^{N-1}\frac{d\sigma}{d\tau}$$

$$\tilde{\sigma}_N(\mu) \sim \int_0^\infty dz\, \frac{z^{N-1}-1}{z} \int_{\mu^2}^{zQ^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2}\left[\overset{\Gamma_{\text{cusp}}}{A(\alpha_s(\bar{\mu}))} + B(\alpha_s(\sqrt{z}Q))\right]$$ Infrared renormalon ambiguity of order Λ_{QCD}^2

Inverse Mellin transformation $\quad \dfrac{d\sigma}{d\tau} \sim \dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} dN\, \tau^{-N}\, \tilde{\sigma}_N$

Comparing with general convention of resummation:

$$\sigma(\omega) \sim \left[1+\sum_{n=1}^{\infty} C_n\left(\frac{\alpha_s}{2\pi}\right)^n\right] e^{\underbrace{L\, g_1(\alpha_s L)}_{LL} + \underbrace{g_2(\alpha_s L)}_{NLL} + \underbrace{\alpha_s g_3(\alpha_s L)}_{NNLL} + \underbrace{\alpha_s^2 g_4(\alpha_s L)}_{N^3LL} + \cdots}$$

$\omega \ll 1$.

$L = \ln\omega \sim \alpha_s^{-1}$

with $\quad g_m(\alpha_s L) = \sum_{n=1}^{\infty} a_{m,n}(\alpha_s L)^n$

We can link the accuracy of resummation to the coefficients of anomalous dimensions.

The terms in the exponents $S(\nu,\mu)$ and $a_\Gamma(\nu,\mu)$ can be splitted into A, B, C, D as follows

A $\quad L(\alpha_s L + \alpha_s^2 L^2 + \cdots)$

B $\quad \alpha_s L + \alpha_s^2 L^2 + \cdots$

$$S(\nu,\mu) = \frac{\Gamma_0}{4\beta_0^2}\Bigg\{\boxed{\frac{4\pi}{\alpha_s(\nu)}\left(1-\frac{1}{r}-\ln r\right)} + \boxed{\left(\frac{\Gamma_1}{\Gamma_0}-\frac{\beta_1}{\beta_0}\right)(1-r+\ln r)+\frac{\beta_1}{2\beta_0}\ln^2 r}$$

LL $\qquad$ NLL

$$+\frac{\alpha_s(\nu)}{4\pi}\left[\left(\frac{\beta_1\Gamma_1}{\beta_0\Gamma_0}-\frac{\beta_2}{\beta_0}\right)(1-r+r\ln r)+\left(\frac{\beta_1^2}{\beta_0^2}-\frac{\beta_2}{\beta_0}\right)(1-r)\ln r - \left(\frac{\beta_1^2}{\beta_0^2}-\frac{\beta_2}{\beta_0}-\frac{\beta_1\Gamma_1}{\beta_0\Gamma_0}+\frac{\Gamma_2}{\Gamma_0}\right)\frac{(1-r)^2}{2}\right]$$

NNLL

$C = \alpha_s(\alpha_s L + \alpha_s^2 L^2 + \cdots)$

N^{3}LL

$$+\left(\frac{\alpha_s(\nu)}{4\pi}\right)^2\Bigg[\left(\frac{\beta_1\beta_2}{\beta_0^2}-\frac{\beta_1^3}{2\beta_0^3}-\frac{\beta_3}{2\beta_0}+\frac{\beta_1}{\beta_0}\left(\frac{\Gamma_2}{\Gamma_0}-\frac{\beta_2}{\beta_0}+\frac{\beta_1^2}{\beta_0^2}-\frac{\beta_1\Gamma_1}{\beta_0\Gamma_0}\right)\frac{r^2}{2}\right)\ln r$$
$$+\left(\frac{\Gamma_3}{\Gamma_0}-\frac{\beta_3}{\beta_0}+\frac{2\beta_1\beta_2}{\beta_0^2}+\frac{\beta_1^2}{\beta_0^2}\left(\frac{\Gamma_1}{\Gamma_0}-\frac{\beta_1}{\beta_0}\right)-\frac{\beta_2\Gamma_1}{\beta_0\Gamma_0}-\frac{\beta_1\Gamma_2}{\beta_0\Gamma_0}\right)\frac{(1-r)^3}{3}$$
$$+\left(\frac{3\beta_3}{4\beta_0}-\frac{\Gamma_3}{2\Gamma_0}+\frac{\beta_1^3}{\beta_0^3}-\frac{3\beta_1^2\Gamma_1}{4\beta_0^2\Gamma_0}+\frac{\beta_2\Gamma_1}{\beta_0\Gamma_0}+\frac{\beta_1\Gamma_2}{4\beta_0\Gamma_0}-\frac{7\beta_1\beta_2}{4\beta_0^2}\right)(1-r)^2$$
$$+\left(\frac{\beta_1\beta_2}{\beta_0^2}-\frac{\beta_3}{\beta_0}-\frac{\beta_1^2\Gamma_1}{\beta_0^2\Gamma_0}+\frac{\beta_1\Gamma_2}{\beta_0\Gamma_0}\right)\frac{1-r}{2}\Bigg] + \underbrace{\cdots}_{\alpha_s^3 g_5(\alpha_s L)}\Bigg\},$$

$D = \alpha_s^2(\alpha_s L + \alpha_s^2 L^2 + \cdots)$

$$a_\Gamma(\nu,\mu) = \frac{\Gamma_0}{2\beta_0}\Bigg\{\boxed{\ln\frac{\alpha_s(\mu)}{\alpha_s(\nu)}} + \boxed{\left(\frac{\Gamma_1}{\Gamma_0}-\frac{\beta_1}{\beta_0}\right)\frac{\alpha_s(\mu)-\alpha_s(\nu)}{4\pi}}$$

NLL B $\qquad$ C NNLL

D

$$+\boxed{\left[\frac{\Gamma_2}{\Gamma_0}-\frac{\beta_2}{\beta_0}-\frac{\beta_1}{\beta_0}\left(\frac{\Gamma_1}{\Gamma_0}-\frac{\beta_1}{\beta_0}\right)\right]\frac{\alpha_s^2(\mu)-\alpha_s^2(\nu)}{32\pi^2}} + \underbrace{\cdots}_{\alpha_s^3 g_5(\alpha_s L)}\Bigg\}$$

N^{3}LL

with

$$A = L\, g_1(\alpha_s L) + g_2^A(\alpha_s L) + \alpha_s g_3^A(\alpha_s L) + \alpha_s^2 g_4^A(\alpha_s L) + \cdots$$

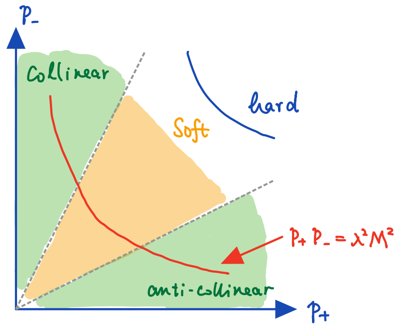
$$B = g_2^B(\alpha_s L) + \alpha_s g_3^B(\alpha_s L) + \alpha_s^2 g_4^B(\alpha_s L) + \cdots$$

$$C = \alpha_s g_3^C(\alpha_s L) + \alpha_s^2 g_4^C(\alpha_s L) + \cdots$$

$$D = \alpha_s^2 g_4^D(\alpha_s L) + \cdots$$

6. TMD factorization & Resummation

• Kinematics



$$N_1(P_1) + N_2(P_2) \to \gamma^*(q) + X \quad , \quad q^2 = M^2$$

$$q^\mu \sim M(1, 1, \lambda) \quad , \quad \lambda = q_T/M \ll 1$$

$$p_c^\mu \sim M(\lambda^2, 1, \underline{\lambda}) \ , \quad p_{\bar c}^\mu \sim M(1, \lambda^2, \underline{\lambda}) \quad , \quad k_s^\mu \sim M(\lambda, \lambda, \underline{\lambda})$$

• Factorization

$$d\sigma = \frac{4\pi\alpha^2}{3q^2 s}\frac{d^4q}{(2\pi)^4}\int d^4x\, e^{-iq\cdot x}(-g_{\mu\nu})\langle N_1(P_1)N_2(P_2)|J^{\nu\dagger}(x)J^\mu(0)|N_1(P_1)N_2(P_2)\rangle$$

$$J^\mu(x) = \bar\psi(x)\gamma^\mu\psi(x)$$

$$\to C_V(-M^2-i0)\,\bar\chi_c(x)\gamma_\perp^\mu\chi_{\bar c}(x) \qquad \text{matching from QCD to SCET}$$

$$\to C_V(-M^2-i0)\,\bar\chi_c(x)S_n^\dagger(x)\gamma_\perp^\mu S_{\bar n}(x)\chi_{\bar c}(x) \qquad \text{decoupling}$$

$$(-g_{\mu\nu})\langle N_1(P_1)N_2(P_2)|J^{\nu\dagger}(x)J^\mu(0)|N_1(P_1)N_2(P_2)\rangle \to -|C_V(-M^2-i0)|^2\frac{1}{N_c}\langle 0|\bar T(S_{\bar n}^\dagger(x)S_n(x))\,T(S_n^\dagger(0)S_{\bar n}(0))|0\rangle$$
$$\times\langle N_1(P_1)N_2(P_2)|[\bar\chi_{\bar c}(x)\gamma_{\perp,\mu}\chi_c(x)][\bar\chi_c(0)\gamma_\perp^\mu\chi_{\bar c}(0)]|N_1(P_1)N_2(P_2)\rangle$$

using Fierz transformation

$$\langle N_1(P_1)N_2(P_2)|[\bar\chi_c(x)\gamma_{\perp,\mu}\chi_{\bar c}(x)][\bar\chi_{\bar c}(0)\gamma_\perp^\mu\chi_c(0)]|N_1(P_1)N_2(P_2)\rangle$$
$$= -\frac12\langle N_2(P_2)|\bar\chi_{\bar c}(0)\gamma^\rho\chi_{\bar c}(x)|N_2(P_2)\rangle\langle N(P_1)|\bar\chi_c(x)\gamma_{\perp,\mu}\gamma_\rho\gamma_\perp^\mu\chi_c(0)|N_1(P_1)\rangle$$
$$-\frac12\langle N_2(P_2)|\bar\chi_{\bar c}(0)\gamma^\rho\gamma_5\chi_{\bar c}(x)|N_2(P_2)\rangle\langle N_1(P_1)|\bar\chi_c(x)\gamma_{\perp,\mu}\gamma_\rho\gamma_5\gamma_\perp^\mu\chi_c(0)|N_1(P_1)\rangle$$

+ zero terms due to $\not{n}\not{n} = \bar{\not{n}}\bar{\not{n}} = 0$

Using following relation

$$\gamma^\rho = g^{\rho\sigma}\gamma_\sigma = \left(g_\perp^{\rho\sigma} + \frac{n^\rho\bar n^\sigma + \bar n^\rho n^\sigma}{2}\right)\cdot\gamma_\sigma = \gamma_\perp^\rho + \frac{\bar{\not{n}}}{2}n^\rho + \frac{\not{n}}{2}\bar n^\rho$$

$$\gamma_\perp^\mu\not{n} = g_\perp^{\mu\nu}\gamma_\nu\not{n} = g_\perp^{\mu\nu}(-\not{n}\gamma_\nu + 2n_\nu) = -\not{n}\gamma_\perp^\mu$$

$$\gamma_{\perp,\mu}\gamma_\perp^\mu = g_\perp^{\mu\nu}\gamma_\mu\gamma_\nu = \frac12 g_\perp^{\mu\nu}\{\gamma_\mu,\gamma_\nu\} = g_\perp^{\mu\nu}g_{\mu\nu}\,\mathbb{1} = \left[g^{\mu\nu}g_{\mu\nu} - \frac12(n\cdot\bar n + n\cdot\bar n)\right]\mathbb{1} = (d-2)\cdot\mathbb{1}$$

$$\Rightarrow \langle N_1(P_1)N_2(P_2)|[\bar\chi_c(x)\gamma_{\perp,\mu}\chi_{\bar c}(x)][\bar\chi_{\bar c}(0)\gamma_\perp^\mu\chi_c(0)]|N_1(P_1)N_2(P_2)\rangle$$
$$= \frac12\langle N_2(P_2)|\bar\chi_{\bar c}(0)\frac{\bar{\not{n}}}{2}\chi_{\bar c}(x)|N_2(P_2)\rangle\, n^\rho\langle N_1(P_1)|\bar\chi_c(x)\frac{\not{n}}{2}\gamma_{\perp,\mu}\gamma_\perp^\mu\chi_c(0)|N_1(P_1)\rangle\,\bar n_\rho$$
$$\underline{-\frac12\langle N_2(P_2)|\bar\chi_{\bar c}(0)\frac{\bar{\not{n}}}{2}\gamma_5\chi_{\bar c}(x)|N_2(P_2)\rangle\, n^\rho\langle N_1(P_1)|\bar\chi_c(x)\frac{\not{n}}{2}\gamma_{\perp,\mu}\gamma_5\gamma_\perp^\mu\chi_c(0)|N_1(P_1)\rangle\,\bar n_\rho} \quad \text{vanish due to } \mathrm{Tr}(\not{n}\bar{\not{n}}\gamma_5) = 0$$
$$= (d-2)\langle N_1(P_1)|\bar\chi_c(x)\frac{\bar{\not{n}}}{2}\chi_c(0)|N_1(P_1)\rangle\langle N_2(P_2)|\bar\chi_{\bar c}(0)\frac{\not{n}}{2}\chi_{\bar c}(x)|N_2(P_2)\rangle$$

$$d\sigma \sim d^4q \, \underbrace{|C_V(-M^2-i0)|^2}_{=H(-M^2-i0)} \int d^4x \, e^{-iq\cdot x} \frac{1}{N_c} \langle 0| \bar{T}(S_{\bar{n}}^\dagger(x) S_n(x)) T(S_n^\dagger(0) S_{\bar{n}}(0)) |0\rangle$$

$$\times \langle N_1(P_1)| \bar{\chi}_c(0) \frac{\bar{n}\!\!\!/}{2} \chi_c(x) |N_1(P_1)\rangle \langle N_2(P_2)| \bar{\chi}_{\bar{c}}(x) \frac{n\!\!\!/}{2} \chi_{\bar{c}}(0) |N_2(P_2)\rangle$$

The soft and (anti-) collinear fields need to multi-pole expansion

$$q^\mu \sim \sqrt{s}(1, 1, \lambda) \Rightarrow x^\mu \sim \frac{1}{\sqrt{s}}(1, 1, \lambda^{-1})$$

$$S_n(x) = S_n(0) + [(\tfrac{x_-}{2} n + \tfrac{x_+}{2}\bar{n} + x_\perp)\cdot \partial S_n](0) + \frac{1}{2}[(\tfrac{x_-}{2} n^\mu + \tfrac{x_+}{2}\bar{n}^\mu + x_\perp^\mu)(\tfrac{x_-}{2} n^\nu + \tfrac{x_+}{2}\bar{n}^\nu + x_\perp^\nu)\partial_\mu\partial_\nu S_n](0)$$

$$= S_n(x_\perp) + \mathcal{O}(\lambda S_n)$$

$$S_{DY}(x_\perp) = \frac{1}{N_c}\langle 0|\bar{T}(S_{\bar{n}}^\dagger(x_\perp) S_n(x_\perp)) T(S_n^\dagger(0) S_{\bar{n}}(0))|0\rangle$$

$$\chi_c(x_-\tfrac{n^\mu}{2} + x_+\tfrac{\bar{n}^\mu}{2} + x_\perp^\mu) = \chi_c(0) + [(\tfrac{x_-}{2} n + \tfrac{x_+}{2}\bar{n} + x_\perp)\cdot\partial\chi_c](0) + \frac{1}{2}[(\tfrac{x_-}{2} n^\mu + \tfrac{x_+}{2}\bar{n}^\mu + x_\perp^\mu)(\tfrac{x_-}{2} n^\nu + \tfrac{x_+}{2}\bar{n}^\nu + x_\perp^\nu)\partial_\mu\partial_\nu\chi_c](0)$$

$$= \chi_c(0) + [(\underbrace{\tfrac{x_+}{2}}_{\mathcal{O}(1)}\underbrace{\bar{n}\cdot\partial}_{\mathcal{O}(1)} + \underbrace{x_\perp}_{\mathcal{O}(\lambda^{-1})}\cdot\underbrace{\partial}_{\mathcal{O}(\lambda)})\chi_c](0) + \frac{1}{2}[(\tfrac{x_+}{2}\bar{n}^\mu + x_\perp^\mu)(\tfrac{x_+}{2}\bar{n}^\nu + x_\perp^\nu)\partial_\mu\partial_\nu\chi_c](0) + \cdots$$

$$= \chi_c(\tfrac{x_+}{2}\bar{n} + x_\perp) + \mathcal{O}(\lambda^2\chi_c)$$

$$\langle N_1(P_1)|\bar{\chi}_c(x)\frac{\bar{n}\!\!\!/}{2}\chi_c(0)|N_1(P_1)\rangle = \langle N_1(P_1)|\bar{\chi}_c(\tfrac{x_+}{2}\bar{n} + x_\perp)\frac{\bar{n}\!\!\!/}{2}\chi_c(0)|N_1(P_1)\rangle$$

$$= (\bar{n}\cdot P_1)\int dz' \, e^{iz' x_+ \bar{n}\cdot P_1/2} B_{q/N_1}(z', x_T^2)$$

$$\Rightarrow \int dx_+ \, e^{-izx_+(\bar{n}\cdot P_1)/2}\langle N_1(P_1)|\bar{\chi}_c(\tfrac{x_+}{2}\bar{n} + x_\perp)\frac{\bar{n}\!\!\!/}{2}\chi_c(0)|N_1(P_1)\rangle = (2\pi)(\bar{n}\cdot P_1)\int dz' \, \delta(z'\tfrac{\bar{n}\cdot P_1}{2} - z\tfrac{\bar{n}\cdot P_1}{2}) B(z', x_T^2)$$

$$\Rightarrow B_{q/N}(z, x_T^2, \mu) = \frac{1}{2\pi}\int d\frac{x_+}{2} \, e^{-izx_+(\bar{n}\cdot P_1)/2}\langle N_1(P_1)|\bar{\chi}_c(\tfrac{x_+}{2}\bar{n} + x_\perp)\frac{\bar{n}\!\!\!/}{2}\chi_c(0)|N_1(P_1)\rangle$$

$$\Rightarrow d\sigma \sim d^4q \, H(-M^2-i0)\int d^4x \, e^{-iq\cdot x}\int dz_1 \, e^{iz_1 x_+(\bar{n}\cdot P_1)/2}\int dz_2 \, e^{iz_2 x_-(n\cdot P_2)/2} B_{q/N_1}(z_1, x_T^2) B_{\bar{q}/N_2}(z_2, x_T^2) S_{DY}(x_\perp)$$

$$= d^4q \, H(-M^2-i0)\int d^2x_\perp \, e^{-iq_\perp\cdot x_\perp} S_{DY}(x_\perp) \, \frac{1}{2}\int dx_+\int dx_- \, e^{-ix_+(\bar{n}\cdot q)/2} e^{-ix_-(n\cdot q)/2}$$

$$\times\int dz_1 \, e^{iz_1 x_+(\bar{n}\cdot P_1)/2}\int dz_2 \, e^{iz_2 x_-(n\cdot P_2)/2} B_{q/N_1}(z_1, x_T^2) B_{\bar{q}/N_2}(z_2, x_T^2)$$

$$= d^4q \, H(-M^2-i0) \, \frac{1}{2}\int d^2x_\perp \, e^{-iq_\perp\cdot x_\perp} S_{DY}(x_\perp)\int_0^1 dz_1\int_0^1 dz_2 \, (2\pi)\delta(\bar{n}\cdot q - z_1\bar{n}\cdot P_1)(2\pi)\delta(n\cdot q - z_2 n\cdot P_2)$$

$$\times(\bar{n}\cdot P_1) B_{q/N_1}(z_1, x_T^2)(n\cdot P_2) B_{\bar{q}/N_2}(z_2, x_T^2)$$

$$\Rightarrow \frac{d\sigma}{dP_T \, dM^2 dY} \sim \frac{1}{4}\int d\phi \, dq_+ dq_- dq_T^2 \, \underbrace{\delta(q^2 - M^2)}_{\delta(q_+q_- - q_T^2 - M^2)} \delta(Y - \tfrac{1}{2}\ln\tfrac{q_-}{q_+}) \delta(q_T^2 - P_T^2)\int_0^1 dz_1\int_0^1 dz_2 \, \delta(z_1 - \tfrac{q_-}{\sqrt{s}}) \delta(z_2 - \tfrac{q_+}{\sqrt{s}})$$

invariant mass and rapidity of virtual photon

$$\times H(-M^2-i0)\int_0^\infty dx_T \, x_T\int d\varphi \, e^{iP_T\cdot x_T\cos\varphi} B_{q/N_1}(z_1, x_T^2) B_{\bar{q}/N_2}(z_2, x_T^2) S_{DY}(x_\perp)$$

$$\left.\begin{array}{l}\frac{q_-}{q_+} = e^{2Y} \\ q_+q_- = \underbrace{M^2 + q_T^2}_{M_T^2}\end{array}\right\} \Rightarrow \left\{\begin{array}{l} q_+ = M_T e^{-Y} \\ q_- = M_T e^{Y}\end{array}\right\} \Rightarrow dq_+ dq_- = \begin{vmatrix}\frac{M}{M_T}e^{-Y} & \frac{M}{M_T}e^{Y} \\ -M_T e^{-Y} & M_T e^{Y}\end{vmatrix} dM\,dY = 2M\,dM\,dY = dM^2 dY$$

$$\Rightarrow dM^2 dY \, \delta(q^2 - M^2)\delta(Y - \tfrac{1}{2}\ln\tfrac{q_-}{q_+}) = dq_+ dq_- \, \delta(q_+ - M_T e^{-Y})\delta(q_- - M_T e^{Y})$$

$$\Rightarrow \frac{d\sigma}{dP_T\, dM^2\, dY} \sim \int_0^1 dz_1 \int_0^1 dz_2\, \delta\!\left(z_1 - \frac{M_T e^{Y}}{\sqrt{s}}\right) \delta\!\left(z_2 - \frac{M_T e^{-Y}}{\sqrt{s}}\right)$$

$$\times\, H(-M^2 - i0) \int_0^\infty dx_T\, x_T \underbrace{\int_0^\pi d\varphi\, e^{i P_T \cdot x_T \cdot \cos\varphi}}_{\text{Bessel function}}\, B_{q/N_1}(z_1, x_T^2)\, B_{\bar q/N_2}(z_2, x_T^2)\, S_{DY}(x_\perp)$$

$$\boxed{\Rightarrow \frac{d\sigma}{dP_T\, dM^2\, dY} = H(-M^2 - i0) \int_0^\infty dx_T\, x_T\, \pi J_0(x_T P_T)\, B_{q/N_1}\Big(\underbrace{\frac{M_T e^{Y}}{\sqrt{s}}}_{\xi_1}, x_T^2\Big)\, B_{\bar q/N_2}\Big(\underbrace{\frac{M_T e^{-Y}}{\sqrt{s}}}_{\xi_2}, x_T^2\Big)\, S_{DY}(x_\perp) + (q \leftrightarrow \bar q)}$$

• Soft function at NLO

$$S_{DY}(x_\perp) = \frac{1}{N_c} \langle 0 | \bar T \left(S_{\bar n}^\dagger(x_\perp)\, S_n(x_\perp) \right) T \left(S_n^\dagger(0)\, S_{\bar n}(0) \right) | 0 \rangle$$

At LO, $S^{(0)}_{DY}(x_\perp) = 1$

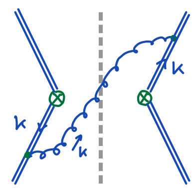
At NLO,

$$S^{(1)}_{DY}(x_\perp) = 2 \int \frac{d^d k}{(2\pi)^d}\, e^{i \vec k_\perp \cdot \vec x_\perp}\, (2\pi)\, \delta(k^2)\, g_s^2\, (-C_F)\, (-g_{\mu\nu})\, \frac{n^\mu \bar n^\nu}{(n\cdot k)(\bar n \cdot k)}$$

$$= 2\, \frac{e^{\varepsilon \gamma_E}}{(4\pi)^\varepsilon}\, \frac{C_F g_s^2}{(2\pi)^{d-1}}\, \frac{2\pi^{\frac{1}{2}-\varepsilon}}{\Gamma(\frac{1}{2}-\varepsilon)}\, \frac{1}{2} \int_0^\infty \frac{dk_+}{k_+} \frac{dk_-}{k_-} \int_0^\infty dk_T^2\, (k_T^2)^{-\varepsilon}\, \delta(k_+ k_- - k_T^2)\, \underbrace{\int_0^\pi d\phi\, \sin^{-2\varepsilon}\phi\, e^{i k_T x_T \cos\phi}}_{=\, \pi^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-\varepsilon)}{\Gamma(1-\varepsilon)}\, {}_0F_1\left(1-\varepsilon;\, -\frac{k_T^2 x_T^2}{4}\right)}$$

Hypergeometric function

$$S^{(1)}_{DY}(x_\perp) = \frac{C_F \alpha_s}{\pi}\, \frac{e^{\varepsilon \gamma_E}}{\Gamma(1-\varepsilon)}\, \underbrace{\int_0^\infty \frac{dk_+}{k_+}}_{\text{ill-defined due to rapidity divergence}}\, \underbrace{\int_0^\infty dk_T^2\, (k_T^2)^{-1-\varepsilon}\, {}_0F_1\left(1-\varepsilon;\, -\frac{k_T^2 x_T^2}{4}\right)}_{\text{well defined}}$$



So we need a regulator to handle rapidity divergence:

$$e^{-ik\cdot x} = e^{-i \frac{k_+ x_- + k_- x_+}{2} + i \vec k_\perp \cdot \vec x_\perp}$$

$$= e^{-b_0 \tau k_0}\, e^{i \vec k_\perp \cdot \vec x_\perp} \quad \Leftarrow \quad x^\mu = (x_+, x_-, \vec x_\perp) = (-i b_0 \tau,\, -i b_0 \tau,\, \vec x_\perp) \qquad b_0 = 2 e^{-\gamma_E}$$

Then NLO soft function can be rewritten as

$$S^{(1)}_{DY}(x_\perp) = \frac{C_F \alpha_s}{\pi}\, \frac{e^{\varepsilon \gamma_E}}{\Gamma(1-\varepsilon)} \int_0^\infty dk_T^2\, (k_T^2)^{-1-\varepsilon}\, {}_0F_1\left(1-\varepsilon;\, -\frac{k_T^2 x_T^2}{4}\right) \int_0^\infty \frac{dk_+}{k_+}\, e^{-\frac{b_0 \tau}{2}\left(k_+ + \frac{k_T^2}{k_+}\right)}$$

$$\int_0^\infty \frac{dk_+}{k_+}\, e^{-\frac{b_0\tau}{2}\left(k_+ + \frac{k_T^2}{k_+}\right)} = \int_0^{k_T} \frac{dk_+}{k_+}\, e^{-\frac{b_0\tau}{2}\frac{k_T^2}{k_+}} + \int_0^{k_T} \frac{dk_+}{k_+} \left[e^{-\frac{b_0\tau}{2}\left(k_+ + \frac{k_T^2}{k_+}\right)} - e^{-\frac{b_0\tau}{2}\frac{k_T^2}{k_+}} \right] + \int_{k_T}^\infty \frac{dk_+}{k_+}\, e^{-\frac{b_0\tau}{2} k_+} + \int_{k_T}^\infty \frac{dk_+}{k_+} \left[e^{-\frac{b_0\tau}{2}\left(k_+ + \frac{k_T^2}{k_+}\right)} - e^{-\frac{b_0\tau}{2} k_+} \right]$$

$$= 2 \int_{k_T}^\infty \frac{dk_+}{k_+}\, e^{-\frac{b_0\tau}{2} k_+} + 2 \int_{k_T}^\infty \frac{dk_+}{k_+}\, e^{-\frac{b_0\tau}{2} k_+} \left(e^{-\frac{b_0\tau}{2}\frac{k_T^2}{k_+}} - 1 \right)$$

$$= 2\, \Gamma\!\left(0, \frac{b_0\tau}{2} k_T\right) + 2 \int_{k_T}^\infty \frac{dk_+}{k_+}\, e^{-\frac{b_0\tau}{2} k_+} \left(-\frac{b_0\tau}{2}\frac{k_T^2}{k_+} + \cdots \right) \qquad \Gamma(s,x) = \int_x^\infty dt\, t^{s-1} e^{-t}$$

$$= 2\, \underline{\Gamma\!\left(0, \frac{b_0\tau}{2} k_T\right)} + 2 \left(-\frac{b_0^2 \tau^2}{4} k_T^2 \right) \underline{\Gamma\!\left(-1, \frac{b_0\tau}{2} k_T\right)} \quad \text{upper incomplete gamma function}$$

$$= -2 \ln(\tau k_T) + O(\tau)$$

$$\Rightarrow \quad S^{(1)}_{DY}(x_\perp) = \frac{C_F\alpha_s}{\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\int_0^\infty dk_T^2\,(k_T^2)^{-1-\varepsilon}\,{}_0F_1\left(1-\varepsilon;-\frac{k_T^2x_T^2}{4}\right)\left[-2\ln(\tau k_T)\right]$$

using relation:

$$\int_0^\infty dy\; y^{-1-\varepsilon-\eta}\,{}_0F_1(1-\varepsilon;-ay) = a^{\eta+\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(-\eta-\varepsilon)}{\Gamma(1+\eta)}$$

Then we have

$$\int_0^\infty dk_T^2\,(k_T^2)^{-1-\varepsilon}\,{}_0F_1\left(1-\varepsilon;-\frac{x_T^2}{4}k_T^2\right) = \left(\frac{x_T^2}{4}\right)^{\varepsilon}\Gamma(1-\varepsilon)\Gamma(-\varepsilon)$$

$$\int_0^\infty dk_T^2\,(k_T^2)^{-1-\varepsilon}\,(-\ln k_T^2)\,{}_0F_1\left(1-\varepsilon;-\frac{x_T^2}{4}k_T^2\right) = \partial_\eta\left[\left(\frac{x_T^2}{4}\right)^{\eta+\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(-\eta-\varepsilon)}{\Gamma(1+\eta)}\right]\Bigg|_{\eta=0}$$

$$= \left(\frac{x_T^2}{4}\right)^{\varepsilon}\Gamma(1-\varepsilon)\Gamma(-\varepsilon)\left[\ln\frac{x_T^2}{4} - \psi^{(0)}(-\varepsilon) + \gamma_E\right]$$

$$\Rightarrow \quad S^{(1)}_{DY}(x_\perp) = \frac{C_F\alpha_s}{\pi}\left(\frac{x_T^2\mu^2}{4e^{-2\gamma_E}}\right)^{\varepsilon}e^{-\varepsilon\gamma_E}\,\Gamma(-\varepsilon)\left[-2\ln\tau + \ln\frac{x_T^2}{4} - \psi^{(0)}(-\varepsilon) + \gamma_E\right]$$

$$= \frac{C_F\alpha_s}{\pi}\left(\frac{x_T^2\mu^2}{4e^{-2\gamma_E}}\right)^{\varepsilon}\left[\frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\left(\frac{x_T^2\tau^{-2}}{4e^{-2\gamma_E}}\right) - \frac{\pi^2}{12}\right]$$

$$\Rightarrow \quad S^{(1)}_{DY}(x_\perp) = \frac{C_F\alpha_s}{4\pi}\left[\frac{4}{\varepsilon^2} + \frac{4}{\varepsilon}\ln(\mu^2\tau^2) - 2\ln^2\frac{x_T^2\mu^2}{b_0^2} + 4\ln\frac{x_T^2\mu^2}{b_0^2}\ln(\mu^2\tau^2) - \frac{\pi^2}{3}\right]$$

- Beam function at NLO

$$B_{q/N}(z,x_T^2,\mu) = \frac{1}{2\pi}\int d\frac{x_+}{2}\,e^{-izx_+(\bar n\cdot P_1)/2}\langle \underline{N_1(P_1)}|\,\bar\chi_c\left(\frac{x_+}{2}\bar n + x_\perp\right)\frac{\not{\bar n}}{2}\chi_c(0)\,|\underline{N_1(P_1)}\rangle$$

hadron external state, non-pertubative

$x_T q_T \sim 1$, for large transverse momenta in perturbative domain, $q_T^2 \gg \Lambda_{QCD}^2$ then $x_T \ll \Lambda_{QCD}^{-1}$. Then beam functions obey an operator-product expansion (OPE)

$$B_{i/N}(z,x_T^2,\mu) = \sum_j\int_z^1\frac{du}{u}\underbrace{I_{i\leftarrow j}(u,x_T^2,\mu)}_{\text{perturbative calculable}}\;\underbrace{\phi_{j/N}(z/u,\mu)}_{\text{parton distribution functions (PDFs)}} + \mathcal{O}(\Lambda_{QCD}^2x_T^2)$$

non-perturbative

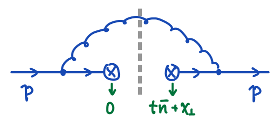
$I_{i\leftarrow j}$ can be extracted by matching, i.e.

$$I^{(0)}_{i\leftarrow j}(z,x_T^2,\mu) = \delta(1-z)\,\delta_{ij}$$

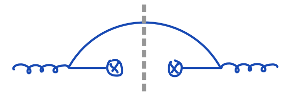
First, we calculate the following matrix elements at NLO

Sample of diagram

$$B_{q/q}(z,x_T^2,\mu) = \frac{1}{2\pi}\int dt\;e^{-izt(\bar n\cdot p)}\langle q(p)|\bar\chi_c\left(\frac{x_+}{2}\bar n + x_\perp\right)\frac{\not{\bar n}}{2}\chi_c(0)|q(p)\rangle$$



$$B_{q/g}(z,x_T^2,\mu) = \frac{1}{2\pi}\int dt\;e^{-izt(\bar n\cdot p)}\langle g(p)|\bar\chi_c\left(\frac{x_+}{2}\bar n + x_\perp\right)\frac{\not{\bar n}}{2}\chi_c(0)|g(p)\rangle$$



We extract $I_{i\leftarrow j}$ from partonic level "PDFs"

$$\underbrace{B_{i/q}(z,x_T^2,\mu)}_{\text{perturbative calculable}} = \sum_j \int_z^1 \frac{du}{u} I_{i\leftarrow j}(u,x_T^2,\mu)\,\underbrace{\phi_{j/q}(z/u,\mu)}_{\text{perturbative calculable}}$$

$$B_{i/q}(z,x_T^2) = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n B_{i/q}^{(n)}, \qquad \phi_{i/q}(z) = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n \phi_{i/q}^{(n)}, \qquad I_{i\leftarrow j}(z,x_T^2) = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n I_{i\leftarrow j}^{(n)}$$

$$B_{i/q}^{(0)}(z,x_T^2) = \delta(1-z)\,\delta_{iq}, \qquad \phi_{i/q}^{(0)}(z) = \delta(1-z)\,\delta_{iq}, \qquad I_{q\leftarrow q}^{(0)}(z,x_T^2) = \delta(1-z)$$

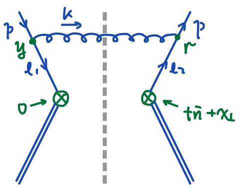
$$B_{i/q}^{(0)}(z,x_T^2,\mu) = \sum_j \int_z^1 \frac{dz}{z} I_{i\leftarrow j}^{(0)}(u,x_T^2,\mu)\,\phi_{j/q}^{(0)}(z/u,\mu)$$

$$B_{i/q}^{(1)}(z,x_T^2,\mu) = \sum_j \int_z^1 \frac{dz}{z} I_{i\leftarrow j}^{(0)}(u,x_T^2,\mu)\,\phi_{j/q}^{(1)}(z/u,\mu) + \sum_j \int_z^1 \frac{dz}{z} I_{i\leftarrow j}^{(1)}(u,x_T^2,\mu)\,\phi_{j/q}^{(0)}(z/u,\mu)$$

Then $I_{i\leftarrow j}$ can be determined by matching between $B_{i/q}$ and $\phi_{j/q}$ order by order in α_s

NLO calculation. starting from definition of the following matrix element:

$$B_{q/q}(z,x_T^2,\mu) = \frac{1}{2\pi}\int dt\, e^{-izt(\bar n\cdot p)} \langle q(p)|(\bar\xi_n W_n)(t\bar n + x_\perp)\underbrace{\frac{\bar{\not n}\not n}{4}\frac{\bar{\not n}}{2}\frac{\not n\bar{\not n}}{4}}_{=\frac{\bar{\not n}}{2}}(W_n^\dagger \xi_n)(0)|q(p)\rangle$$



diag. a

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-izt(\bar n\cdot p)} \int d^d y \int d^d r \int \frac{d^d k}{(2\pi)^d} e^{-ik(r-y)} \frac{-ig_{\mu\nu}}{k^2+i0} \int \frac{d^d l_1}{(2\pi)^d} e^{-il_1(0-y)} \int \frac{d^d l_2}{(2\pi)^d} e^{-il_2[r-(t\bar n + x_\perp)]}$$

$$\text{color average} \to \times \frac{1}{N_c}\, \bar u(p)\, e^{ip\cdot r}\, ig_s \gamma^\mu t^a_{ij} \frac{i\not l_2}{l_2^2+i0}\frac{\bar{\not n}}{2}\frac{i\not l_1}{l_1^2+i0}\, ig_s\gamma^\nu t^a_{ji}\, e^{-ip\cdot y} u(p)$$

$$= \underbrace{g_s^2 \int \frac{d^d k}{(2\pi)^d}\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{-izt(\bar n\cdot p)} e^{i(p-k)(t\bar n + x_\perp)} \frac{i}{k^2+i0}}_{\text{Phase-space } \int d\Pi_c(z,x_T^2)}\ \underbrace{\frac{1}{N_c}\left[(-g_{\mu\nu})\,\bar u(p)\, i\gamma^\mu t^a_{ij} \frac{i(\not p-\not k)}{(p-k)^2}\frac{\bar{\not n}}{2}\frac{i(\not p-\not k)}{(p-k)^2}\, i\gamma^\nu t^a_{ij}\, u(p)\right]}_{\text{Amplitude } \mathcal{M}_{\text{diag.}a}}$$

$$\int d\Pi_c(z,x_T^2) = g_s^2 \int \frac{d^d k}{(2\pi)^d}\,\delta[z\bar n\cdot p - \bar n\cdot(p-k)]\, e^{-ik\cdot x_\perp}(2\pi)\,\delta(k^2)$$

$\frac{1}{k^2+i0} \to (-2\pi i)\,\delta(k^2)$, $\quad p^\mu = (\bar n\cdot p)\frac{n^\mu}{2} \Rightarrow p\cdot x_\perp = 0$

$$= \frac{g_s^2}{(2\pi)^{d-1}}\frac{1}{4}\int_0^\infty dk_+ \int_0^\infty dk_- \int_0^\infty dk_T^2\,(k_T^2)^{-\varepsilon}\,\delta(k_T^2 - k_+k_-)\,\delta(k_- - (1-z)\bar n\cdot p)\,\frac{2\pi^{\frac{1}{2}-\varepsilon}}{\Gamma(\frac{1}{2}-\varepsilon)}\int_0^\pi d\phi\,\sin^{-2\varepsilon}\phi\, e^{ik_T\cdot x_T\cos\phi}$$

$$= \frac{\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\int_0^\infty dk_T^2\,\frac{(k_T^2)^{-\varepsilon}}{(1-z)\,\bar n\cdot p}\; {}_0F_1\!\left(1-\varepsilon;\,-\frac{k_T^2 x_T^2}{4}\right)$$

$$\mathcal{M}_{\text{diag.}a} = (-g_{\mu\nu})\,C_F\,\frac{1}{2}\,\mathrm{tr}\!\left[\not p\,\gamma^\mu(\not p-\not k)\frac{\bar{\not n}}{2}(\not p-\not k)\gamma^\nu\right]$$

$\gamma^\nu\gamma^\mu\gamma_\nu = (2-d)\gamma^\mu$

$(p-k)^2 = -2p\cdot k = -(\bar n\cdot p)\,k_+$

$\mathrm{tr}(\not k\frac{\bar{\not n}}{2}) + \mathrm{tr}(\frac{\bar{\not n}}{2}\not k) = 4\bar n\cdot k \Rightarrow \mathrm{tr}(\not k\frac{\bar{\not n}}{2}) = 2\bar n\cdot k$

$$= C_F\,\frac{1}{[(p-k)^2]^2}\,\frac{1}{2}(d-2)\,\mathrm{tr}\!\left[\not p\,(\not p-\not k)\frac{\bar{\not n}}{2}(\not p-\not k)\right]$$

$$= C_F(d-2)\,\frac{k_-}{(\bar n\cdot p)k_+}$$

with $k_+ = k_T^2/k_-$, $k_- = (1-z)\,\bar n\cdot p$ by constraints from phase space

Thus, diagram a gives contribution as

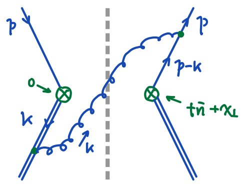
$$\int d\Pi_c(z,x_T^2)\,\mathcal{M}_{\text{diag.}a} = \frac{C_F\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\left(\frac{x_T^2\mu^2}{b_0^2}\right)^{\varepsilon}(1-z)\,2(1-\varepsilon)\left(-\frac{1}{\varepsilon} - \frac{\pi^2}{6}\varepsilon + \mathcal{O}(\varepsilon^2)\right)$$

where the following two integrations have been used:

$$\frac{2\pi^{\frac{1}{2}-\epsilon}}{\Gamma(\frac{1}{2}-\epsilon)}\int_0^{\pi} d\phi\, \sin^{-2\epsilon}\phi\; e^{i k_T\cdot x_T\cos\phi} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}\,{}_0F_1\left(1-\epsilon;-\frac{k_T^2x_T^2}{4}\right)$$

$$\int_0^{\infty} dk_T^2\,(k_T^2)^{-1-\epsilon}\,{}_0F_1\left(1-\epsilon;-\frac{k_T^2x_T^2}{4}\right) = -e^{-2\epsilon\gamma_E}\,\epsilon\,\Gamma^2(-\epsilon)\left(\frac{x_T^2}{4e^{-2\gamma_E}}\right)^{\epsilon} = \left(\frac{x_T^2}{b_0^2}\right)^{\epsilon}\left(-\frac{1}{\epsilon}-\frac{\pi^2}{6}\epsilon+O(\epsilon^2)\right)$$

The corresponding contribution to $\phi_{q/q}(z)$ can be obtained by setting $x_T=0$ straightforwardly. It leads to zero contribution because of factor $\left(\frac{x_T^2\mu^2}{b_0^2}\right)^{\epsilon}$. In another word, it is scaleless.



diag. b

$$\mathcal{M}_{diag.b} = \frac{1}{N_c}(-g_{\mu\nu})\left[\bar{u}(p)\, i\gamma^{\mu}t^a_{ij}\,\frac{i(\not{p}-\not{k})}{(p-k)^2}\,\frac{\not{\bar{n}}}{2}\,u(p)\right] i t^a_{ji}\,\frac{i\bar{n}^{\nu}}{\bar{n}\cdot k}$$

$$= C_F\,\frac{1}{(\bar{n}\cdot p)k_+k_-}\,\frac{1}{2}\,tr\left[\not{p}\,\not{\bar{n}}\,(\not{p}-\not{k})\,\frac{\not{\bar{n}}}{2}\right]$$

$$= 2C_F\,\frac{\bar{n}\cdot(p-k)}{k_+k_-}$$

Combine with the phase space integration, we have

$$\int d\Pi_c(z,x_T^2)\,\mathcal{M}_{diag.b} = \frac{\alpha_s}{4\pi}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\int_0^{\infty}dk_T^2\,\frac{(k_T^2)^{-\epsilon}}{(1-z)\,\bar{n}\cdot p}\,{}_0F_1\left(1-\epsilon;-\frac{k_T^2x_T^2}{4}\right)\times 2C_F\frac{\bar{n}\cdot(p-k)}{k_+k_-}\Bigg|_{\substack{k_+=k_T^2/k_-\\ k_-=(1-z)\bar{n}\cdot p}}$$

$$= \frac{C_F\alpha_s}{4\pi}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\int_0^{\infty}dk_T^2\,(k_T^2)^{-1-\epsilon}\,{}_0F_1\left(1-\epsilon;-\frac{k_T^2x_T^2}{4}\right)\frac{2z}{1-z} \leftarrow$$

singular when $z\to 1$. need to be regulated

To resolve the poles around $z\to 1$ (or $k_-\to 0$), we insert regulator $e^{-\frac{b_0\tau}{2}\left(k_-+\frac{k_T^2}{k_-}\right)}$.

To express $\frac{1}{k_-}e^{-\frac{b_0\tau}{2}\left(k_-+\frac{k_T^2}{k_-}\right)}$ as $\delta(k_-)$ and $\left(\frac{1}{k_-}\right)_+$, the following technique should be introduced:

$$\lim_{\tau\to 0}\frac{1}{x}e^{-\frac{\tau}{x}} = \left(\frac{1}{x}\right)_*^{[\tau e^{\gamma_E}]}$$

which can be derived as follows:

$$\int_0^a dx\,\frac{1}{x}e^{-\frac{\tau}{x}}f(x) = \int_0^{a/\tau}\frac{du}{u}e^{-\frac{1}{u}}\left[f(\tau u)-f(0)\right] + f(0)\int_0^{a/\tau}\frac{du}{u}e^{-\frac{1}{u}}$$

$$= \int_0^{\infty}dt\left(\frac{1}{t}\right)_+e^{-t} + O(\tau) = -\gamma_E + O(\tau)$$

$$= \int_{\tau/a}^{\infty}\frac{dt}{t}e^{-t}\left[f\left(\frac{\tau}{t}\right)-f(0)\right] + f(0)\left[\overbrace{\int_{\tau/a}^{1}\frac{dt}{t}(e^{-t}-1)+\int_1^{\infty}\frac{dt}{t}e^{-t}} + \underbrace{\int_{\tau/a}^{1}\frac{dt}{t}}_{=\ln a-\ln\tau}\right]$$

$$= \underbrace{\int_{\tau/a}^{1}\frac{dt}{t}(e^{-t}-1)\left[f\left(\frac{\tau}{t}\right)-f(0)\right]}_{O(\tau\ln\tau)} + \underbrace{\int_1^{\infty}\frac{dt}{t}e^{-t}\left[f\left(\frac{\tau}{t}\right)-f(0)\right]}_{O(\tau)} + \underbrace{\int_{\tau/a}^{1}\frac{dt}{t}\left[f\left(\frac{\tau}{t}\right)-f(0)\right]}_{=\int_0^a\frac{dx}{x}[f(x)-f(0)]+O(\tau)} + \underbrace{f(0)\left(-\gamma_E+\ln a-\ln\tau\right)}_{=f(0)\int_{\tau e^{\gamma_E}}^{a}\frac{dx}{x}}$$

$$= \int_0^a\frac{dx}{x}\left[f(x)-f(0)\right] + f(0)\int_{\tau e^{\gamma_E}}^{a}\frac{dx}{x}$$

$$= \int_0^a dx\left[\left(\frac{1}{x}\right)_*^{[\tau e^{\gamma_E}]} + O(\tau\ln\tau)\right]f(x)$$

To prove $\lim_{\tau\to0}\int_{\tau/a}^{1}\frac{dt}{t}(e^{-t}-1)\left[f(\frac{\tau}{t})-f(0)\right]=0$, we use $g(\frac{\tau}{t})=f(\frac{\tau}{t})-f(0)$ and do expansion

$$\int_{\tau/a}^{1}\frac{dt}{t}(e^{-t}-1)\,g(\frac{\tau}{t}) = \int_\tau^a\frac{du}{u}(e^{-\frac{\tau}{u}}-1)\,g(u) = \int_\tau^a du\sum_{n=1}\frac{1}{n!}\frac{(-\tau)^n}{u^{n+1}}\,g(u) = \sum_{n=1}\frac{1}{n!}(-\tau)^n\int_\tau^a\frac{(du^{-n})}{-n}\,g(u)$$

$$= -\sum_{n=1}\frac{1}{n!\cdot n}\left\{\left[(-\frac{\tau}{a})^n g(a)-(-1)^n g(\tau)\right]-(-\tau)^n\int_\tau^a du\,u^{-n}g'(u)\right\}$$

$$= -\sum_{n=1}\frac{1}{n!\cdot n}\left\{\left[(-\frac{\tau}{a})^n g(a)-(-1)^n g(\tau)\right]-(-\tau)^{n+1}\frac{1}{-n}\int_\tau^a du^{-n}\,g'(u)\right\}-\tau\int_\tau^a d\ln u\,g'(u)$$

$$= -\sum_{n=1}\frac{1}{n!\cdot n}\left\{\left[(-\frac{\tau}{a})^n g(a)-(-1)^n \underbrace{g(\tau)}_{\text{vanish as }\tau\to0}\right]-\frac{\tau}{n}\left[(-\frac{\tau}{a})^n g'(a)-(-1)^n g'(\tau)\right]-\underbrace{\frac{(-\tau)^{n+1}}{n}\int_\tau^a du\,g''(u)\,u^{-n}}_{O(\tau^2\ln\tau)}\right\}$$

$$+\underbrace{\tau\ln\tau\,g'(u)}_{O(\tau\ln\tau)}-\tau\ln a\,g'(u)$$

So we can see $\lim_{\tau\to0}\int_{\tau/a}^{1}\frac{dt}{t}(e^{-t}-1)\left[f(\frac{\tau}{t})-f(0)\right]=0$

Then we can do expansion for combination of rapidity regulator and $1/k_-$

$$\lim_{k_-\to0} e^{-\frac{b_0\tau}{2}(k_-+\frac{k_T^2}{k_-})}\frac{1}{k_-} \simeq e^{-\frac{b_0\tau}{2}\frac{k_T^2}{k_-}}\frac{1}{k_-} = \left(\frac{1}{k_-}\right)_*^{[e^{-\gamma_E}\tau k_T^2 e^{\gamma_E}]} = \left(\frac{1}{k_-}\right)_*^{[\tau k_T^2]}$$

Now we change variable from $k_-\to(1-z)\bar{n}\cdot p$

$$\int_0^a dk_-\left(\frac{1}{k_-}\right)_*^{[\tau k_T^2]}f(k_-) = \int_0^a\frac{dk_-}{k_-}\left[f(k_-)-f(0)\right]+f(0)\int_{\tau k_T^2}^{a}\frac{dk_-}{k_-}$$

$$= \int_\xi^1\frac{dz}{1-z}\left[f((1-z)\bar{n}\cdot p)-f(0)\right]+f(0)\int_\xi^{1-\frac{\tau k_T^2}{\bar{n}\cdot p}}\frac{dz}{1-z}\,,\quad \text{with } \xi=1-\frac{a}{\bar{n}\cdot p}$$

$$= \int_\xi^1 dz\left(\frac{1}{1-z}\right)_+ f((1-z)\bar{n}\cdot p)+f(0)\int_0^{1-\frac{\tau k_T^2}{\bar{n}\cdot p}}\frac{dz}{1-z}$$

$$\Rightarrow \int_\xi^1 dz\,(\bar{n}\cdot p)\left(\frac{1}{k_-}\right)_*^{[\tau k_T^2]}f(k_-) = \int_\xi^1 dz\left[\left(\frac{1}{1-z}\right)_+ +\delta(1-z)\cdot\ln\frac{\bar{n}\cdot p}{\tau k_T^2}\right]f((1-z)\bar{n}\cdot p)$$

$$\Rightarrow \lim_{k_-\to0} e^{-\frac{b_0\tau}{2}(k_-+\frac{k_T^2}{k_-})}\frac{1}{k_-} = \frac{1}{\bar{n}\cdot p}\left[\left(\frac{1}{1-z}\right)_+ +\delta(1-z)\cdot\ln\frac{\bar{n}\cdot p}{\tau k_T^2}\right]$$

With the exponential regulator, the contribution from diag.b can be calulated as follows:

$$\int d\Pi_c(z,x_T^2)\,\mathcal{M}_{diag.b} = \frac{C_F\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\int_0^\infty dk_T^2\,(k_T^2)^{-1-\varepsilon}\,{}_0F_1(1-\varepsilon;-\frac{k_T^2x_T^2}{4})\,2z\,(\bar{n}\cdot p)\,e^{-\frac{b_0\tau}{2}(k_-+\frac{k_T^2}{k_-})}\frac{1}{k_-}$$

$$= \frac{C_F\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\int_0^\infty dk_T^2\,(k_T^2)^{-1-\varepsilon}\,{}_0F_1(1-\varepsilon;-\frac{k_T^2x_T^2}{4})\,2z\left[\left(\frac{1}{1-z}\right)_+ +\delta(1-z)\cdot\ln\frac{\bar{n}\cdot p}{\tau k_T^2}\right]$$

$$= \frac{C_F\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\left(\frac{x_T^2}{4e^{-2\gamma_E}}\right)^{\varepsilon}\left\{\left[2\ln\frac{\bar{n}\cdot p}{\tau}\delta(1-z)+2z\left(\frac{1}{1-z}\right)_+\right]\left(-\frac{1}{\varepsilon}-\frac{\pi^2}{6}\varepsilon+O(\varepsilon^2)\right)+2\delta(1-z)\left[\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{x_T^2}{4e^{-2\gamma_E}}+O(\varepsilon)\right]\right\}$$

$$= \frac{C_F\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\left(\frac{x_T^2\mu^2}{b_0^2}\right)^{\varepsilon}\left\{\left[\frac{2}{\varepsilon^2}-\frac{2}{\varepsilon}\ln\frac{x_T^2(\bar{n}\cdot p)}{b_0^2\tau}\right]\delta(1-z)-\frac{2}{\varepsilon}\frac{z}{(1-z)_+}\right\}$$

$$= \frac{C_F\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\left\{\left[\frac{2}{\varepsilon^2}-\frac{2}{\varepsilon}\ln\frac{\bar{n}\cdot p}{\tau\mu^2}\right]\delta(1-z)-\frac{2}{\varepsilon}\frac{z}{(1-z)_+}+\delta(1-z)\left[\ln^2\frac{x_T^2\mu^2}{b_0^2}-2\ln\frac{x_T^2(\bar{n}\cdot p)}{b_0^2\tau}\ln\frac{x_T^2\mu^2}{b_0^2}\right]-\frac{2z}{(1-z)_+}\ln\frac{x_T^2\mu^2}{b_0^2}\right\}$$

$$\Rightarrow \int d\Pi_c(z,x_T^2)\, \mathcal{M}_{diag.b} = \frac{C_F\alpha_s}{4\pi}\left\{\left[\frac{2}{\varepsilon^2} - \frac{2}{\varepsilon}\ln\frac{\bar{n}\cdot p}{\tau\mu^2}\right]\delta(1-z) - \frac{2}{\varepsilon}\frac{z}{(1-z)_+}\right.$$

$$\left. + \delta(1-z)\left(-\ln^2\frac{x_T^2\mu^2}{b_0^2} - 2\ln\frac{\bar{n}\cdot p}{\tau\mu^2}\ln\frac{x_T^2\mu^2}{b_0^2} - \frac{\pi^2}{6}\right) - \frac{2z}{(1-z)_+}\ln\frac{x_T^2\mu^2}{b_0^2}\right\}$$

Again, the corresponding contribution to $\phi_{q/q}(z)$ can be obtained by setting $x_T=0$ straightforwardly, which is scaleless.

There are non-vanish contribution to beam function from soft-collinear overlap region!

Zero-bin : Collinear matrix elements in soft limit $(k_+, k_-, k_T) \sim M(\lambda,\lambda,\lambda)$
account for overlap in soft-collinear region

Phase-space of collinear sector in soft limit

$$\int d\Pi_c(z,x_T^2) = \frac{g_s^2}{(2\pi)^{d-1}}\frac{1}{4}\int_0^\infty dk_+\int_0^\infty dk_-\int_0^\infty dk_T^2\,(k_T^2)^{-\varepsilon}\delta(k_T^2-k_+k_-)\,\delta[k_- - (1-z)\bar{n}\cdot p]\,\frac{2\pi^{\frac{1}{2}-\varepsilon}}{\Gamma(\frac{1}{2}-\varepsilon)}\int_0^\pi d\phi\,\sin^{-2\varepsilon}\phi\, e^{ik_T\cdot x_T\cos\phi}$$

$$\xrightarrow[\text{limit}]{\text{soft}} \frac{1}{\bar{n}\cdot p}\delta(1-z)\frac{g_s^2}{(2\pi)^{d-1}}\frac{1}{4}\int_0^\infty dk_+\int_0^\infty dk_-\int_0^\infty dk_T^2\,(k_T^2)^{-\varepsilon}\delta(k_T^2-k_+k_-)\frac{2\pi^{\frac{1}{2}-\varepsilon}}{\Gamma(\frac{1}{2}-\varepsilon)}\int_0^\pi d\phi\,\sin^{-2\varepsilon}\phi\, e^{ik_T\cdot x_T\cos\phi}$$

Amplitudes of collinear sector in soft limit

$$\mathcal{M}_{diag.a} = C_F(d-2)\frac{k_-}{(\bar{n}\cdot p)k_+} \quad \text{invariant in soft limit} \sim M^{-1}\mathcal{O}(1) \leftarrow \text{Power Suppressed}$$

$$\mathcal{M}_{diag.b} = 2C_F\frac{\bar{n}\cdot(p-k)}{k_+k_-} \xrightarrow[\text{limit}]{\text{soft}} C_F(\bar{n}\cdot p)\frac{\bar{n}\cdot n}{k_+k_-} \quad \sim M^{-1}\mathcal{O}(\lambda^{-2})$$

So the contribution to $B_{q/q}(z,x_T^2)$ from zero-bin region is

$$B_{q/q}^{(1),0b}(z,x_T^2) = \int d\Pi_c^{0b}(z,x_T^2)\left(\mathcal{M}_{diag.a}^{0b} + 2\mathcal{M}_{diag.b}^{0b}\right)$$

$$= \delta(1-z)\,2\,\frac{C_F g_s^2}{(2\pi)^{d-1}}\frac{1}{4}\int_0^\infty dk_+\int_0^\infty dk_-\int_0^\infty dk_T^2\,(k_T^2)^{-\varepsilon}\delta(k_T^2-k_+k_-)\frac{2\pi^{\frac{1}{2}-\varepsilon}}{\Gamma(\frac{1}{2}-\varepsilon)}\int_0^\pi d\phi\,\sin^{-2\varepsilon}\phi\, e^{ik_T\cdot x_T\cos\phi}\frac{\bar{n}\cdot n}{k_+k_-}$$

$$= \delta(1-z)\, S_{DY}^{(1)}(x_T)$$

Since the corrections to $\phi_{q/q}(z)$ at NLO and beyond are zero (the same for the other chanels, e.g. $q\leftarrow g$)

$$\phi_{i/j}(z) = \delta(1-z)$$

Here $\phi_{i/j}(z)$ refers to partonic PDFs at bare level, we can also define the renormalized by

$$\phi_{i/j}(z,\mu) = \sum_k\left[Z_{i\leftarrow k}^{-1}(\mu)\otimes\phi_{k/j}\right](z)$$

See 1908.03831

then $I_{q\leftarrow q}^{(1)}(z,x_T^2)$ is totally the same with $B_{q/q}^{(1)}(z,x_T^2)$

$$I_{q\leftarrow q}^{(1)}(z,x_T^2) = B_{q/q}^{(1)}(z,x_T^2)$$

$$= \int d\Pi_c(z,x_T^2)\left(\mathcal{M}_{diag.a} + 2\mathcal{M}_{diag.b}\right) - B_{q/q}^{(1),0b}(z,x_T^2)$$

$$= \frac{C_F\alpha_s}{4\pi}\left\{\frac{1}{\varepsilon}\left[\left(-4\ln\frac{\bar{n}\cdot p}{\tau\mu^2} - 4\ln\mu^2\tau^2\right)\delta(1-z) - 2(1-z) - \frac{4z}{(1-z)_+}\right]\right.$$

$$\left. + \delta(1-z)\left[-4\left(\ln\frac{\bar{n}\cdot p}{\tau\mu^2} + \ln(\mu^2\tau^2)\right)\ln\frac{x_T^2\mu^2}{b_0^2}\right] + \left[-2(1-z) - \frac{4z}{(1-z)_+}\right]\ln\frac{x_T^2\mu^2}{b_0^2} + 2(1-z)\right\}$$

$$\int_a^1 dz\left[(1-z)+\frac{2z}{(1-z)_+}\right]f(z) = \int_a^1 dz\,\frac{1}{1-z}\left[2z\,f(z)-2f(1)\right] - 2f(1)\int_0^a dz\,\frac{1}{1-z} + \int_a^1 dz\,(1-z)\left[f(z)-f(1)\right] + f(1)\int_a^1 dz\,(1-z)$$

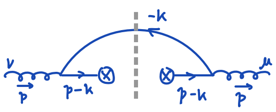
$$= \int_a^1 dz\left(\frac{1+z^2}{1-z}\right)_+ f(z) + f(1)\int_0^1 dz\left[\frac{2z-2}{1-z}+(1-z)\right] = \int_a^1 dz\left[\left(\frac{1+z^2}{1-z}\right)_+ - \frac{3}{2}\delta(1-z)\right]f(z)$$

$$\Rightarrow \quad (1-z)+\frac{2z}{(1-z)_+} = \left(\frac{1+z^2}{1-z}\right)_+ - \frac{3}{2}\delta(1-z)$$

$$\Rightarrow \quad I^{(1)}_{q\leftarrow q}(z,x_T^2) = \frac{C_F\alpha_s}{4\pi}\left\{\frac{1}{\varepsilon}\left[\delta(1-z)\left(-4\ln(\tau\bar{n}\cdot p)+3\right) - 2\left(\frac{1+z^2}{1-z}\right)_+\right]\right.$$

$$\left. + \delta(1-z)\left[\left(-4\ln(\tau\bar{n}\cdot p)+3\right)\ln\frac{x_T^2\mu^2}{b_0^2}\right] - 2\left(\frac{1+z^2}{1-z}\right)_+ \ln\frac{x_T^2\mu^2}{b_0^2} + 2(1-z)\right\}$$

To complete quark beam function, $q\leftarrow g$ channel should also be taken into account



diag. c

$$\mathcal{M}_{diag.c} = \underbrace{\frac{1}{N_c^2-1}\,\frac{1}{d-2}}_{\text{spin average}}\left(-g_{\mu\nu}+\frac{\bar{n}_\mu p_\nu+\bar{n}_\nu p_\mu}{\bar{n}\cdot p}\right)\underbrace{(-1)}_{\text{anti-commute of fermion}}\mathrm{tr}\left[i\gamma^\mu t^a_{ij}\frac{i(\not{p}-\not{k})}{(p-k)^2}\frac{\not{\bar{n}}}{2}\frac{i(\not{p}-\not{k})}{(p-k)^2}i\gamma^\nu t^a_{ji}(-\not{k})\right]$$

$$= T_F\,\frac{1}{(\bar{n}\cdot p)^2 k_T^2}\,\frac{1}{d-2}\left\{-\mathrm{tr}\left[(\not{p}-\not{k})\frac{\not{\bar{n}}}{2}(\not{p}-\not{k})\gamma^\mu\not{k}\gamma_\mu\right] + \frac{1}{\bar{n}\cdot p}\mathrm{tr}\left[(\not{p}-\not{k})\frac{\not{\bar{n}}}{2}(\not{p}-\not{k})\not{\bar{n}}\not{k}\not{p}\right]\right.$$

$$\left. + \frac{1}{\bar{n}\cdot p}\mathrm{tr}\left[(\not{p}-\not{k})\frac{\not{\bar{n}}}{2}(\not{p}-\not{k})\not{p}\not{k}\not{\bar{n}}\right]\right\}$$

using the following Dirac algebra:

$$-\mathrm{tr}\left[(\not{p}-\not{k})\frac{\not{\bar{n}}}{2}(\not{p}-\not{k})\gamma^\mu\not{k}\gamma_\mu\right] = (d-2)\,\mathrm{tr}\left[\frac{\not{\bar{n}}}{2}\not{p}\not{k}\not{p}\right] = (d-2)(\bar{n}\cdot p)k_+\,\mathrm{tr}\left(\frac{\not{\bar{n}}}{2}\not{p}\right) = 2(d-2)(\bar{n}\cdot p)\,k_T^2/(1-z)$$

$$\frac{1}{\bar{n}\cdot p}\mathrm{tr}\left[(\not{p}-\not{k})\frac{\not{\bar{n}}}{2}(\not{p}-\not{k})\not{\bar{n}}\not{k}\not{p}\right] = 2z\,\mathrm{tr}\left[\frac{\not{\bar{n}}}{2}\not{k}\not{p}(-\not{k})\right] = -2z(\bar{n}\cdot p)\cdot k_+\,\mathrm{tr}\left(\frac{\not{\bar{n}}}{2}\not{k}\right) = -4z(\bar{n}\cdot p)\,k_T^2$$

$$\frac{1}{\bar{n}\cdot p}\mathrm{tr}\left[(\not{p}-\not{k})\frac{\not{\bar{n}}}{2}(\not{p}-\not{k})\not{p}\not{k}\not{\bar{n}}\right] = 2z\,\mathrm{tr}\left[\frac{\not{\bar{n}}}{2}(-\not{k})\not{p}\not{k}\right] = -4z(\bar{n}\cdot p)\,k_T^2$$

$$\gamma^\nu\gamma^\mu\gamma_\nu = (2-d)\gamma^\mu$$
$$\mathrm{tr}(\not{\bar{n}}\cdot\not{k}) = 4\bar{n}\cdot k$$

$$\mathcal{M}_{diag.c} = T_F\,\frac{\bar{n}\cdot p(1-z)}{k_T^2}\,\frac{2}{1-\varepsilon}\left[z^2+(1-z)^2-\varepsilon\right]$$

$$B^{(1)}_{q\leftarrow g}(z,x_T^2) = \int d\Pi_c(z,x_T^2)\,\mathcal{M}_{diag.c}$$

$$= \frac{\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\int_0^\infty dk_T^2\,\frac{(k_T^2)^{-\varepsilon}}{(1-z)\,\bar{n}\cdot p}\;{}_0F_1\!\left(1-\varepsilon;-\frac{k_T^2x_T^2}{4}\right)T_F\,\frac{\bar{n}\cdot p(1-z)}{k_T^2}\,\frac{2}{1-\varepsilon}\left[z^2+(1-z)^2-\varepsilon\right]$$

$$= \frac{\alpha_s}{4\pi}T_F\left(\frac{x_T^2}{b_0^2}\right)^{\varepsilon}\left(-\frac{1}{\varepsilon}-\frac{\pi^2}{6}\varepsilon+\mathcal{O}(\varepsilon^2)\right)\frac{2}{1-\varepsilon}\left[z^2+(1-z)^2-\varepsilon\right]$$ ← no zero-bin for soft quark

Again, $\phi^{(1)}_{q\leftarrow g}$ is scaleless once with set $x_T^2=0$. So we can get:

$$I^{(1)}_{q\leftarrow g}(z,x_T^2) = B^{(1)}_{q\leftarrow g}(z,x_T^2) = \frac{\alpha_s}{4\pi}T_F\left\{2\left[z^2+(1-z)^2\right]\left(-\frac{1}{\varepsilon}-\ln\frac{x_T^2\mu^2}{b_0^2}\right)+4z(1-z)+\mathcal{O}(\varepsilon)\right\}$$

To see the full cancellation of poles, we need take the renormalization of standard PDFs into account, which is given by

$$\phi_{i/N}(z) = \sum_j\left[Z_{i\leftarrow j}(\mu)\otimes\phi_{j/N}(\mu)\right](z)$$

$$Z_{i\leftarrow j}(\xi,\mu) = \delta(1-\xi)\,\delta_{ij} + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}Z^{[1,1]}_{ij}(\xi,\mu)$$

For convenience, we have defined symbol $\otimes$ for convolution of two functions $f(z)$ and $g(z)$

$$(f\otimes g)(z) \equiv \int_z^1\frac{dw}{w}\,f\!\left(\frac{z}{w}\right)g(w)$$

This operation has two properties:

1. Commutative : $(f\otimes g)(z) = (g\otimes f)(z)$

2. associative : $[f\otimes(g\otimes h)](z) = [(f\otimes g)\otimes h](z)$

Proof of the first :

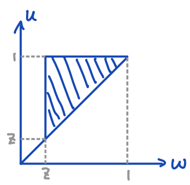
$$(f\otimes g)(z) \equiv \int_z^1 \frac{dw}{w} f\left(\frac{z}{w}\right) g(w) = \int_z^1 \frac{du}{u} f(u)\, g\left(\frac{z}{u}\right)$$

$w = z/u$

$$= (g\otimes f)(z)$$

Proof of the second:

$$[f\otimes(g\otimes h)](z) = \int_z^1 \frac{dw}{w} f\left(\frac{z}{w}\right)\left[\int_w^1 \frac{du}{u} g\left(\frac{w}{u}\right) h(u)\right] = \int_z^1 \frac{du}{u} h(u) \int_z^u \frac{dw}{w} f\left(\frac{z}{w}\right) g\left(\frac{w}{u}\right)$$



$$= \int_z^1 \frac{du}{u} h(u) \int_{z/u}^1 \frac{dw}{w} f\left(\frac{z/u}{w}\right) g(w) = \int_z^1 \frac{dt}{t} h\left(\frac{z}{t}\right) \int_t^1 \frac{dw}{w} f\left(\frac{t}{w}\right) g(w)$$

$$= [h\otimes(f\otimes g)](z) = [(f\otimes g)\otimes h](z)$$

To relate the anomalous dimension and Z-factor. we need to introduce inverse Z-factor. Because the Z-factor for PDFs is non-local and in matrix form of $\mathbb{Z}$. we first study the inverse transformation for the Z-factor matrix, which is defined as

$$\left[\mathbb{Z}^{-1}\otimes\left(\mathbb{Z}\otimes\vec{\phi}_{*/N}(\mu)\right)\right](z) = \vec{\phi}_{*/N}(z)$$

$$\Rightarrow \quad (\mathbb{Z}^{-1}\otimes\mathbb{Z})(z) = \mathbb{1}\cdot\delta(1-z)$$

The above identity can be written in the following form explictly

$$\sum_j \int_z^1 \frac{dw}{w} Z^{-1}_{i\leftarrow j}\left(\frac{z}{w},\mu\right) Z_{j\leftarrow k}(w,\mu) = \delta(1-z)\,\delta_{ik}$$

Then the inverse Z-factor at NLO can be derived as

$$\mathbb{Z}(z) = \mathbb{1}\,\delta(1-z) + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\underline{\mathbb{Z}^{[1,1]}(z)} + \mathcal{O}(\alpha_s^2)$$

Coefficient of $1/\varepsilon$ at $\mathcal{O}(\alpha_s)$

$$\Rightarrow \quad \mathbb{Z}^{-1}(z) = \mathbb{1}\,\delta(1-z) - \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\mathbb{Z}^{[1,1]}(z) + \mathcal{O}(\alpha_s^2)$$

Becaue the bare PDFs $\phi_{i/N}(z)$ don't depend on μ. we have

$$\left[\left(\frac{d}{d\ln\mu}\mathbb{Z}\right)\otimes\vec{\phi}_{*/N}(\mu)\right](z) + \left[\mathbb{Z}\otimes\left(\frac{d}{d\ln\mu}\vec{\phi}_{*/N}(\mu)\right)\right](z) = 0$$

$$\frac{d}{d\ln\mu}\vec{\phi}_{*/N}(z,\mu) = \left[-\mathbb{Z}^{-1}\otimes\left(\frac{d}{d\ln\mu}\mathbb{Z}\right)\otimes\vec{\phi}_{*/N}(\mu)\right](z)$$

$$= \left[2\alpha_s\frac{\partial}{\partial\alpha_s}\mathbb{Z}^{[1]}\otimes\vec{\phi}_{*/N}(\mu)\right](z)$$

Coefficient of single pole $1/\varepsilon$ to all order in α_s

Compare with RGE of the standard parton distribution functions (PDFs), which obey DGLAP evolution:

$$\frac{d}{d\ln\mu}\phi_{i/N}(z,\mu) = \sum_j \left[P_{i\leftarrow j}(\mu)\otimes\phi_{j/N}(\mu)\right](z)$$

we have the renormalization for standard PDFs at NLO

Coefficient of $1/\varepsilon$ at $O(\alpha_s)$

$$\underline{Z_{ij}^{[1,1]}}(z) = \frac{1}{2}P^{(1)}_{i\leftarrow j}(z,\mu)$$

where

$$P^{(1)}_{q\leftarrow q}(z) = 4C_F\left(\frac{1+z^2}{1-z}\right)_+ , \qquad P^{(1)}_{q\leftarrow g}(z) = 4T_F\left[z^2+(1-z)^2\right]$$

$$P^{(1)}_{g\leftarrow q}(z) = 4C_F\,\frac{1+(1-z)^2}{z}, \qquad P^{(1)}_{g\leftarrow g}(z) = 8C_A\left[\frac{z}{(1-z)_+}+\frac{1-z}{z}+z(1-z)\right]+2\beta_0\,\delta(1-z)$$

Finally, the bare beam function at NLO is given by

$$B^{(1)}_{q/N}(z,x_T^2) = \left(I_{q\leftarrow q}(x_T^2)\otimes\phi_{q/N}\right)(z) + \left(I_{q\leftarrow g}(x_T^2)\otimes\phi_{g/N}\right)(z)$$

with

$$\left(I_{q\leftarrow q}\otimes\phi_{q/N}\right)(z) = \phi_{q/N}(z,\mu)+\frac{\alpha_s}{4\pi}\left[\left(I^{(1)}_{q\leftarrow q}+\frac{1}{\varepsilon}Z^{[1,1]}_{qq}\right)\otimes\phi_{q/N}(\mu)\right](z)$$

$$= \phi_{q/N}(z,\mu)+\frac{C_F\alpha_s}{4\pi}\left\{\left[\frac{1}{\varepsilon}\left(-4\ln(\tau\bar n\cdot p)+3\right)+\left(-4\ln(\tau\bar n\cdot p)+3\right)\ln\frac{x_T^2\mu^2}{b_0^2}\right]\underline{\phi_{q/N}(z,\mu)}\right.$$

$$\left.+\int_z^1\frac{du}{u}\left[-2\left(\frac{1+u^2}{1-u}\right)_+\ln\frac{x_T^2\mu^2}{b_0^2}+2(1-u)\right]\underline{\phi_{q/N}\left(\frac{z}{u},\mu\right)}\right\}$$

renormalized PDFs free of divergences

$$\left(I_{q\leftarrow g}\otimes\phi_{g/N}\right)(z) = \frac{\alpha_s}{4\pi}\left[\left(I^{(1)}_{q\leftarrow g}+\frac{1}{\varepsilon}Z^{[1,1]}_{qg}\right)\otimes\phi_{g/N}(\mu)\right](z)$$

$$= \frac{T_F\alpha_s}{4\pi}\int_z^1\frac{du}{u}\left[-2\left[u^2+(1-u)^2\right]\ln\frac{x_T^2\mu^2}{b_0^2}+4u(1-u)\right]\underline{\phi_{g/N}\left(\frac{z}{u},\mu\right)}$$

The pole of ε in the bare quark beam function is

$$B^{(1)}_{q/N}(z,x_T^2)\Big|_{\text{poles}} = \frac{C_F\alpha_s}{4\pi}\,\frac{1}{\varepsilon}\left[-4\ln(\tau\bar n\cdot p)+3\right]\phi_{q/N}(z,\mu)$$

Combining with the bare soft function :

$$S^{(1)}_{DY}(x_\perp) = \frac{C_F\alpha_s}{4\pi}\left[\frac{4}{\varepsilon^2}+\frac{4}{\varepsilon}\ln(\mu^2\tau^2)-2\ln^2\frac{x_T^2\mu^2}{b_0^2}+4\ln\frac{x_T^2\mu^2}{b_0^2}\ln(\mu^2\tau^2)-\frac{\pi^2}{3}\right]$$

The poles can be written as

$$B_{q/N_1}(\xi_1,x_T^2)\,B_{\bar q/N_2}(\xi_2,x_T^2)\,S_{DY}(x_\perp)\Big|_{\text{poles}} = \phi_{q/N_1}(\xi_1,\mu)\,\phi_{\bar q/N_2}(\xi_2,\mu)\,\frac{\alpha_s}{4\pi}C_F\left[\frac{4}{\varepsilon^2}+\frac{1}{\varepsilon}\left(\underline{-4\ln\frac{\bar n\cdot p_1\,n\cdot p_2}{\mu^2}}+6\right)\right]$$

hard logs, collinear anomaly!

which exactly cancel with the poles in the hard coefficient

$$|C_V(-M^2-i0)|^2 = 1+\frac{C_F\alpha_s}{4\pi}\left[-\frac{4}{\varepsilon^2}+\frac{1}{\varepsilon}\left(4\ln\frac{M^2}{\mu^2}-6\right)-2\ln^2\frac{M^2}{\mu^2}+6\ln\frac{M^2}{\mu^2}+\frac{7\pi^2}{3}-16+O(\varepsilon)\right]$$

- Renormalization & RGE

Here and below, we replace τ with $\nu = 1/\tau$, and write the dependence of ν explicitly.

The renormalization for beam function is local

$$B_{i/N}(z, x_T^2, \mu, \nu) = Z_B(\mu, \nu)\, B_{i/N}(z, x_T^2, \nu)$$

At NLO, the divergence part of $B_{q/N}$ has been known at NLO, so we have

$$Z_B(\mu, \nu) = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon} C_F \left(4 \ln \frac{\bar{n}\cdot p}{\nu} - 3\right)$$

The RGE of beam function is

$$\frac{d}{d\ln\mu} B_{i/N}(z, x_T^2, \mu, \nu) = -\Gamma_\mu^B(\mu,\nu)\, B_{i/N}(z, x_T^2, \mu, \nu)$$

$$\Gamma_\mu^B(\mu,\nu) = 2\alpha_s \frac{\partial}{\partial \alpha_s} Z_B^{[1]} = \frac{\alpha_s}{4\pi} C_F\left(8 \ln\frac{\bar{n}\cdot p}{\nu} - 6\right)$$

$$\Rightarrow \Gamma_\mu^B(\mu,\nu) = 2\,\Gamma_{cusp}(\alpha_s) \ln \frac{z\,\bar{n}\cdot P}{\nu} + 2\gamma_\mu^B(\alpha_s), \quad \text{with } \gamma_0^B = -3C_F$$

Convention is different from 1908.03831

The renormalized splitting kernel $I_{q\leftarrow i}(z, x_T^2, \mu, \nu)$ can be defined as follows:

$$B_{i/N}(z, x_T^2, \mu, \nu) = \sum_j \left[I_{q\leftarrow j}(x_T^2, \mu, \nu) \otimes \phi_{j/N}(\mu)\right](z)$$

Comparing with the definition at bare level, we can relate renormalized $I_{i\leftarrow j}$ to the bare one

$$B_{i/N}(z, x_T^2, \nu) = \sum_j \left[I_{q\leftarrow j}(x_T^2, \nu) \otimes \phi_{j/N}\right](z)$$

$$\Rightarrow B_{i/N}(z, x_T^2, \mu, \nu) = \sum_{j,k} \left[Z_B(\mu,\nu)\, I_{q\leftarrow k}(x_T^2, \nu) \otimes Z_{k\leftarrow j} \otimes \phi_{j/N}(\mu)\right](z)$$

$$\Rightarrow I_{q\leftarrow j}(z, x_T^2, \mu, \nu) = \sum_k Z_B(\mu,\nu) \left[I_{q\leftarrow k}(x_T^2, \nu) \otimes Z_{k\leftarrow j}\right](z)$$

Above relation can be written explicitly

$$I_{q\leftarrow q}(z, x_T^2, \mu, \nu) = Z_B(\mu,\nu)\Big[I^{(0)}_{q\leftarrow q}(x_T^2,\nu) \otimes Z^{(0)}_{q\leftarrow q} + \underbrace{I^{(0)}_{q\leftarrow q}(x_T^2,\nu) \otimes Z^{(1)}_{q\leftarrow q} + I^{(1)}_{q\leftarrow q}(x_T^2,\nu) \otimes Z^{(0)}_{q\leftarrow q}}_{\mathcal{O}(\alpha_s)}$$
$$+ \underbrace{I^{(1)}_{q\leftarrow q}(x_T^2,\nu) \otimes Z^{(1)}_{q\leftarrow q} + \cdots}_{\mathcal{O}(\alpha_s^2)} \Big]$$

$$I_{q\leftarrow g}(z, x_T^2, \mu, \nu) = Z_B(\mu,\nu)\Big[\underbrace{I^{(0)}_{q\leftarrow g}(x_T^2,\nu) \otimes Z^{(1)}_{g\leftarrow g} + I^{(1)}_{q\leftarrow g}(x_T^2,\nu) \otimes Z^{(0)}_{g\leftarrow g}}_{\mathcal{O}(\alpha_s)} + \mathcal{O}(\alpha_s^2)\Big]$$

Then the renormalized splitting kernels up to NLO are given by

$$I_{q\leftarrow q}(z, x_T^2, \mu, \nu) = \delta(1-z) + \frac{C_F\alpha_s}{4\pi}\left[\left(-4\ln\frac{\bar{n}\cdot p}{\nu} + 3\right)\ln\frac{x_T^2\mu^2}{b_0^2}\,\delta(1-z) - 2\left(\frac{1+z^2}{1-z}\right)_+ \ln\frac{x_T^2\mu^2}{b_0^2} + 2(1-z)\right]$$

$$I_{q\leftarrow g}(z, x_T^2, \mu, \nu) = \frac{T_F\alpha_s}{4\pi}\left[-2\left[z^2+(1-z)^2\right]\ln\frac{x_T^2\mu^2}{b_0^2} + 4z(1-z)\right]$$

RGE of $I_{i\leftarrow j}$ can be derived from

$$\frac{d}{d\ln\mu} B_{i/N}(z, x_T^2, \mu, \nu) = \sum_j \left[\left(\frac{d}{d\ln\mu} I_{i\leftarrow j}(x_T^2, \mu, \nu) \right) \otimes \phi_{j/N}(\mu) \right](z) + \sum_j \left[I_{i\leftarrow j}(x_T^2, \mu, \nu) \otimes \frac{d}{d\ln\mu} \phi_{j/N}(\mu) \right](z)$$

$$\Rightarrow \frac{d}{d\ln\mu} I_{i\leftarrow j}(z, x_T^2, \mu, \nu) = \left[-2\Gamma_{cusp}(\alpha_s) \ln\frac{z\bar{n}\cdot P}{\nu} - 2\gamma_\mu^B(\alpha_s) \right] I_{i\leftarrow j}(z, x_T^2, \mu, \nu) - \sum_k \left[I_{i\leftarrow k}(x_T^2, \mu, \nu) \otimes P_{k\leftarrow j} \right](z)$$

Renormalization & RGE for soft function

Renormalization of soft function is given by

$$S_{DY}(x_\perp, \mu, \nu) = Z_S(\mu, \nu)\, S_{DY}(x_\perp)$$

Renormalization factor can be determined from bare soft function:

$$S_{DY}(x_\perp) = 1 + \frac{C_F \alpha_s}{4\pi} \left[\frac{4}{\varepsilon^2} + \frac{4}{\varepsilon} \ln\frac{\mu^2}{\nu^2} - 2\ln^2\frac{x_T^2\mu^2}{b_0^2} + 4\ln\frac{x_T^2\mu^2}{b_0^2} \ln\frac{\mu^2}{\nu^2} - \frac{\pi^2}{3} \right]$$

$$\Rightarrow Z_S(\mu, \nu) = 1 + \frac{C_F \alpha_s}{4\pi} \left(-\frac{4}{\varepsilon^2} - \frac{4}{\varepsilon} \ln\frac{\mu^2}{\nu^2} \right)$$

$$S_{DY}(x_\perp, \mu, \nu) = 1 + \frac{C_F \alpha_s}{4\pi} \left(-2\ln^2\frac{x_T^2\mu^2}{b_0^2} + 4\ln\frac{x_T^2\mu^2}{b_0^2} \ln\frac{\mu^2}{\nu^2} - \frac{\pi^2}{3} \right)$$

The RGE of soft function is

$$\frac{d}{d\ln\mu} S_{DY}(x_\perp, \mu, \nu) = -\Gamma_\mu^S(\mu, \nu)\, S_{DY}(x_\perp, \mu, \nu)$$

$$\Gamma_\mu^S(\mu, \nu) = 2\alpha_s \frac{\partial Z_S^{[1]}}{\partial \alpha_s} = \frac{\alpha_s}{4\pi} \left(-8 C_F \ln\frac{\mu^2}{\nu^2} \right)$$

The genenarl form of $\Gamma_{S,\mu}$ can be written as

$$\Gamma_\mu^S(\mu, \nu) = 2\Gamma_{cusp}(\alpha_s(\mu)) \ln\frac{\nu^2}{\mu^2} + 2\gamma_\mu^S(\alpha_s(\mu))$$

with

$$\gamma_\mu^S(\alpha_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^{n+1} \gamma_{\mu,n}^S \,, \qquad \gamma_{\mu,0}^S = 0$$

The rapidity renomalization group equation (RRGE) is

$$\frac{d}{d\ln\nu} S_{DY}(x_\perp, \mu, \nu) = -\Gamma_\nu^S(\alpha_s, \mu)\, S_{DY}(x_\perp, \mu, \nu)$$

Based on the NLO result, we have

$$\Gamma_\nu^S(\alpha_s, \mu) = \frac{\alpha_s(\mu)}{4\pi} 8 C_F \ln\frac{x_T^2\mu^2}{b_0^2} + o(\alpha_s)$$

It is not difficult to see

$$\frac{d}{d\ln\nu} \Gamma_\mu^S(\mu, \nu) = \frac{d}{d\ln\mu} \Gamma_\nu^S(\alpha_s, \mu) = 4\Gamma_{cusp}(\alpha_s(\mu))$$

The structure of Γ_ν^S is

$$\Gamma_\nu^S \sim \alpha_s \left(L_\mu + \gamma_0^R \right) + \alpha_s^2 \left(L_\mu^2 + L_\mu + \gamma_1^R \right) + \alpha_s^3 \left(L_\mu^3 + L_\mu^2 + L_\mu + \gamma_2^R \right) + \cdots$$

$$\underbrace{L_\mu,\, L_\mu^2,\, L_\mu^3, \ldots}_{\Gamma_0} \qquad \underbrace{L_\mu,\, L_\mu^2, \ldots}_{\Gamma_1} \qquad \underbrace{L_\mu, \ldots}_{\Gamma_2}$$

known up to 3-loop Li, Zhu, 1604.00392

need to be resummed to all order in α_s because each tower contribute to large logs at the same accuracy

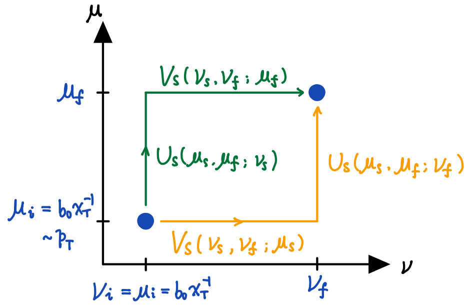
For $\mu \gg x_T^{-1}$, there are large logs in Γ_ν^S, which need to be resummed by

$$\Gamma_\nu^S(\alpha_s, \mu) = \left[4\int_{\mu_i = \frac{b_0}{x_T}}^{\mu} \frac{d\bar{\mu}}{\bar{\mu}} \Gamma_{cusp}(\alpha_s(\mu')) \right] + 4\gamma^R(\alpha_s(\mu_i))$$

Convention is different from 1908.03831

with $\gamma^R(\alpha_s) = \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^{n+1} \gamma_0^R \,, \quad \gamma_0^R = 0$

The evolution of soft function is in 2 dimenion plane, which can be performed through any path connecting (ν_i, μ_i) to (ν_f, μ_f). So we have



$$S_{DY}(x_T^2, \mu_f, \nu_f) = U_S(\mu_s, \mu_f; \nu_f)\, V_S(\nu_s, \nu_f; \mu_s)\, S_{DY}(x_T^2, \mu_s, \nu_s)$$

$$= V_S(\nu_s, \nu_f; \mu_f)\, U_S(\mu_s, \mu_f; \nu_s)\, S_{DY}(x_T^2, \mu_s, \nu_s)$$

Next, we check the above equivalence by evaluting the evolution factors U_s and V_s explicitly.

$$V_S(\nu_s, \nu_f; \mu_s) = \exp\left[-\int_{\nu_s}^{\nu_f} \frac{d\bar\nu}{\bar\nu}\, \Gamma_\nu^S(\alpha_s(\mu_s), \mu_s)\right] = \exp\left\{-\int_{\nu_s}^{\nu_f} \frac{d\bar\nu}{\bar\nu}\left[\underbrace{4\int_{\mu_s}^{\mu_s} \frac{d\bar\mu}{\bar\mu}\, \Gamma_{cusp}(\alpha_s(\bar\mu))}_{=0} + 4\gamma^R(\alpha_s(\mu_s))\right]\right\}$$

$$= \left(\frac{\nu_f^2}{\nu_s^2}\right)^{-2\gamma^R(\alpha_s(\mu_s))}$$

$$U_S(\mu_s, \mu_f; \nu_f) = \exp\left[-\int_{\mu_s}^{\mu_f} \frac{d\bar\mu}{\bar\mu}\, \Gamma_\mu^S(\mu, \nu_f)\right] = \exp\left[-4S_P(\mu_s, \mu_f) + 2a_{\gamma_\mu^S}(\mu_s, \mu_f) + 4a_P(\mu_s, \mu_f)\ln\frac{\nu_f}{\mu_s}\right]$$

$$= \exp\left[-4S_P(\mu_s, \mu_f) + 2a_{\gamma_\mu^S}(\mu_s, \mu_f)\right]\left(\frac{\nu_f^2}{\mu_s^2}\right)^{2a_P(\mu_s, \mu_f)}$$

$$U_S(\mu_s, \mu_f; \nu_s) = \exp\left[-4S_P(\mu_s, \mu_f) + 2a_{\gamma_\mu^S}(\mu_s, \mu_f)\right]\left(\frac{\nu_s^2}{\mu_s^2}\right)^{2a_P(\mu_s, \mu_f)}$$

$$V_S(\nu_s, \nu_f; \mu_f) = \exp\left[-\int_{\nu_s}^{\nu_f} \frac{d\bar\nu}{\bar\nu}\, \Gamma_\nu^S(\alpha_s(\mu_s), \mu_s)\right]$$

$$= \exp\left\{-\int_{\nu_s}^{\nu_f} \frac{d\bar\nu}{\bar\nu}\left[4\int_{\mu_s}^{\mu_f} \frac{d\bar\mu}{\bar\mu}\, \Gamma_{cusp}(\alpha_s(\bar\mu)) + 4\gamma^R(\alpha_s(\mu_s))\right]\right\}$$

$$= \left(\frac{\nu_f^2}{\nu_s^2}\right)^{2a_P(\mu_s, \mu_f) - 2\gamma^R(\alpha_s(\mu_s))}$$

Then it is easy to see that

$$U_S(\mu_s, \mu_f; \nu_f)\, V_S(\nu_s, \nu_f; \mu_s) = V_S(\nu_s, \nu_f; \mu_f)\, U_S(\mu_s, \mu_f; \nu_s)$$

$$= \exp\left[-4S_P(\mu_s, \mu_f) + 2a_{\gamma_\mu^S}(\mu_s, \mu_f)\right]\left(\frac{\nu_f^2}{\mu_s^2}\right)^{2a_P(\mu_s, \mu_f)}\left(\frac{\nu_f^2}{\nu_s^2}\right)^{-2\gamma^R(\alpha_s(\mu_s))}$$

To eliminate the logs in NLO soft function, the initial scales should be chosen as:

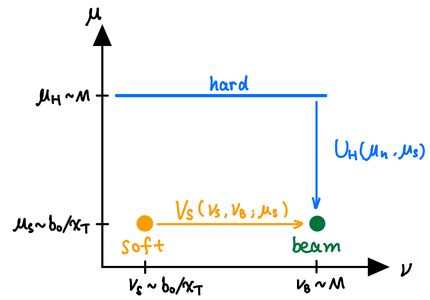
$$\nu_s = \mu_s = \frac{b_0}{x_T}$$

Now we have the evolved soft function

$$S_{DY}(x_T^2, \mu_f, \nu_f) = \exp\left[-4S_P(\mu_s, \mu_f) + 2a_{\gamma_\mu^S}(\mu_s, \mu_f)\right]\left(\frac{x_T^2\nu_f^2}{b_0^2}\right)^{2a_P(\mu_s, \mu_f) - 2\gamma^R(\alpha_s(\mu_s))} S_{DY}(x_T^2, \mu_s, \nu_s)$$

• Resummation

Because the RGE of splitting kernel $I_{i\leftarrow j}(z, x_T^2, \mu)$ is non-local, the easist way to resum large logs is to maintain the beam function at its characteristic scale $(\mu_B, \nu_B) \sim (b_0/x_T, M)$, and to evolve the hard and soft function from their characteristic scales to (μ_B, ν_B), just as shown in the right figure.



First, due to the cancellation of rapidity scale ν, we can maitain the beam function at $(\mu, \nu_B = M)$, and run the rapidity scale in soft function from $\nu_s = \mu_s = b_0/x_T$ to $\nu_B = M$

$$B_{q/N_1}(\xi_1, x_T^2, \mu, \nu_B)\, B_{\bar{q}/N_2}(\xi_2, x_T^2, \mu, \nu_B)\, S_{DY}(x_T^2, \mu, \mu_s) \qquad \xi_1 = \frac{M_T e^{Y}}{\sqrt{s}} \qquad \xi_2 = \frac{M_T e^{-Y}}{\sqrt{s}}$$

$$= \left[I_{q\leftarrow q}(x_T^2, \mu, \nu_B) \otimes \phi_{q/N}(\mu) \right](\xi_1) \left[I_{\bar{q}\leftarrow \bar{q}}(x_T^2, \mu, \nu_B) \otimes \phi_{\bar{q}/N}(\mu) \right](\xi_2)\, S_{DY}(x_T^2, \mu, \nu_B)$$

$$= \exp\left[-4 S_P(\mu_s, \mu) + 2 a_{\gamma_\mu^s}(\mu_s, \mu)\right] \left(\frac{x_T^2 M^2}{b_0^2}\right)^{2a_P(\mu_s,\mu) - 2\gamma^R(\alpha_s(\mu_s))} \qquad 2a_P(\mu_s,\mu) - 2\gamma^R(\alpha_s(\mu_s)) = -F_{q\bar{q}}(x_T, \mu) \leftarrow \text{Collins-Soper kernel}$$

$$\times \Bigg\{ \int_{\xi_1}^1 \frac{dz_1}{z_1} \int_{\xi_2}^1 \frac{dz_2}{z_2}\, \phi_{q/N_1}\!\left(\frac{\xi_1}{z_1}, \mu\right) \phi_{\bar{q}/N_2}\!\left(\frac{\xi_2}{z_2}, \mu\right)$$

$$\times \Bigg(\delta(1-z_1)\delta(1-z_2) + \frac{C_F \alpha_s}{4\pi} \Big[\delta(1-z_1)\delta(1-z_2) \left(6 L_\perp - \frac{\pi^2}{3}\right) \qquad L_\perp = \ln\frac{\mu^2}{\mu_s^2} = \ln\frac{x_T^2\mu^2}{b_0^2}$$

$$+ \left[-2\left(\frac{1+z_1^2}{1-z_1}\right)_+ L_\perp + 2(1-z_1)\right]\delta(1-z_2) + \delta(1-z_1)\left[-2\left(\frac{1+z_2^2}{1-z_2}\right)_+ L_\perp + 2(1-z_2)\right] \Big] \Bigg)$$

$$+ \frac{T_F \alpha_s}{4\pi} \int_{\xi_1}^1 \frac{dz_1}{z_1} \int_{\xi_2}^1 \frac{dz_2}{z_2}\, \phi_{g/N_1}\!\left(\frac{\xi_1}{z_1}, \mu\right) \phi_{\bar{q}/N_2}\!\left(\frac{\xi_2}{z_2}, \mu\right) \delta(1-z_1)\left[-2\left[z_1^2 + (1-z_1)^2\right] L_\perp + 4 z_1(1-z_1)\right]$$

$$+ \frac{T_F \alpha_s}{4\pi} \int_{\xi_1}^1 \frac{dz_1}{z_1} \int_{\xi_2}^1 \frac{dz_2}{z_2}\, \phi_{q/N_1}\!\left(\frac{\xi_1}{z_1}, \mu\right) \phi_{g/N_2}\!\left(\frac{\xi_2}{z_2}, \mu\right) \left[-2\left[z_2^2 + (1-z_2)^2\right] L_\perp + 4 z_2(1-z_2)\right]\delta(1-z_2)$$

$$+ O(\alpha_s^2) \Bigg\}$$

with

$$L_\perp = \ln\frac{\mu^2}{\mu_s^2} = \ln\frac{x_T^2\mu^2}{b_0^2}$$

The Collins – Soper kernel can be expanded as:

$$F_{q\bar{q}}(x_T, \mu) = -2 a_P(\mu_s, \mu) + 2\gamma^R(\alpha_s(\mu_s))$$

$$= \sum_{n=1}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^n d_n$$

with

$$d_1 = \Gamma_0 L_\perp \,,$$

$$d_2 = \frac{\beta_0 \Gamma_0}{2} L_\perp^2 + \Gamma_1 L_\perp + 2\gamma_1^R \,.$$

where we have used the fact $\gamma_0^R = 0$.