Introduction to SCET

Sep. 20 to Oct. 13. 2022, every Tuesday & Thursday, 10am to 12am

Room B1, University of Bern

by Ze Long LIU

References: 1. "Intro. to SCET". arXiv: 1410.1892, by T. Becher, A. Broggio, A. Ferroglia
2. "Lectures on the SCET", by Iain Stewart, (Google it!)
↳ MIT Open Course Ware, YouTube
3. "Lectures on SCET", by M. Beneke in Summer School, Dubna, 2005
4. Textbook "QCD and Collider Physics", by R.K. Ellis, W.J. Stirling and B.R. Webber

• Content

1. Introduction

2. Method of Region
   - ▷ Light-cone decomposition   eikonal approximation

3. Fields and Operators in SCET
   - ▷ Power Counting of fields
   - ▷ Multi-pole expansion

   comparing with method of region

4. Fixed-order Calculation of matrix elements
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5. Renormalization & Renormalization-group (RG) equations
   - ▷ Resummation in Momentum Space   hep-ph/0607228

example:

thrust in e^+e^- collision

hep-ph: 0803.0342

6. TMD factorization: an application in SCET-2
   - ▷ Rapidity regulators
   - ▷ Calculation of Beam function
   - ▷ Rapidity Renormalization Group

example:

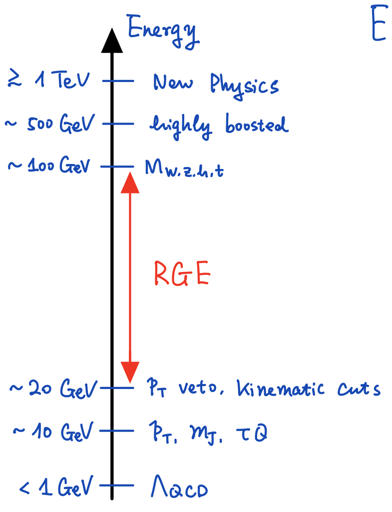
q_T distribution for Drell-Yan

hep-ph: 1202.0814

hep-ph: 1908.03831

1. Introduction

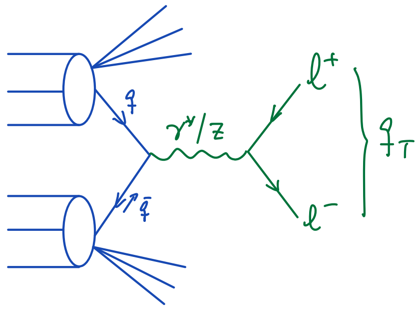
- Multi scales are involved in measurements of collider observables

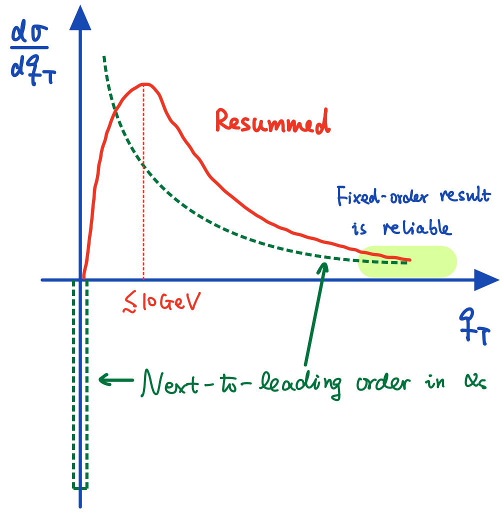


Example:

transverse momentum distribution

for Drell-Yan process





Fixed-Order result

$$\frac{d\sigma}{dq_T} \sim A_0\,\delta(q_T) + \sum_{n=1}^{\infty} \alpha_s^n \left[C_1^n \left(\frac{1}{q_T}\ln^{2n-1}\frac{q_T}{\mu}\right)_* + C_2^n \left(\frac{1}{q_T}\ln^{2n-2}\frac{q_T}{\mu}\right)_* + \cdots \right] + \mathcal{O}(\alpha_s^2)$$

renormalization scale

Definition of star-function

$$\int_0^{p_T} dq_T \left(\frac{1}{q_T}\ln^n\frac{q_T}{\mu}\right)_* f(q_T) = \int_0^{p_T} dq_T\,[f(q_T)-f(0)]\,\frac{1}{q_T}\ln^n\frac{q_T}{\mu} + f(0)\int_\mu^{p_T} dq_T\,\frac{1}{q_T}\ln^n\frac{q_T}{\mu}$$

- Most of the "global" observables studied so far have the property of exponentiation.

$$\sigma(\omega) \sim \left[1 + \sum_{n=1}^{\infty} C_n\left(\frac{\alpha_s}{2\pi}\right)^n\right] e^{\underbrace{L g_1(\alpha_s L)}_{\text{LL}} + \underbrace{g_2(\alpha_s L)}_{\text{NLL}} + \underbrace{\alpha_s g_3(\alpha_s L)}_{\text{NNLL}} + \cdots} + \mathcal{O}(\omega), \qquad \omega \ll 1 . \quad L = \ln\omega \sim \alpha_s^{-1}$$

$$= e^{L g_1(\alpha_s L)}\left[g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \sum_{n=1}^{\infty} C_n\left(\frac{\alpha_s}{2\pi}\right)^n + \cdots\right]$$

distribution is given by $\frac{d\sigma}{d\omega}$ $\qquad e^{\alpha_s L^2 + \alpha_s^2 L^3 + \alpha_s^3 L^4}$

- Accuracy of resummation of large logarithms

	LL	NLL	NNLL		N^{3}LL	
NLO	$\alpha_s L^2$	$\alpha_s L$	α_s			
NNLO	$\alpha_s^2 L^4$	$\alpha_s^2 L^3$	$\alpha_s^2 L^2$	$\alpha_s^2 L$	α_s^2	
N^{3}LO	$\alpha_s^3 L^6$	$\alpha_s^3 L^5$	$\alpha_s^3 L^4$	$\alpha_s^3 L^3$	$\alpha_s^3 L^2$	$\alpha_s^3 L$
	⋮	⋮	⋮	⋮	⋮	⋮

☆ SCET admits scale separation at fields (operator) level!

★ SCET-I

- Observables:

  Jet/Beam Thrust, Jet mass,

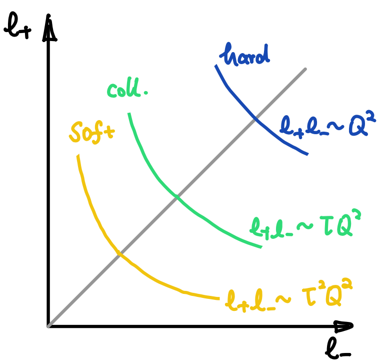
  Threshold limit in DIS, Drell-Yan, $pp \to V+j$

  Inclusive $B \to X_s \gamma$

  ......

- Scale $\quad \ell^\mu = \overset{\ell_+}{(n\cdot\ell)}\frac{\bar{n}^\mu}{2} + \overset{\ell_-}{(\bar{n}\cdot\ell)}\frac{n^\mu}{2} + \ell_\perp^\mu$

	$\ell_+,\ \ell_-,\ \ell_\perp$
Hard:	$Q(1, 1, 1)$
collinear:	$Q(\tau, 1, \sqrt{\tau})$
(ultra-)Soft:	$Q(\tau, \tau, \tau)$



- Dimentional Regularization ($d = 4 - 2\varepsilon$)

  is enough

- RGE

  $\gamma_H \simeq -4\Gamma_{cusp} \ln\frac{\mu}{Q}$

  $\left.\begin{array}{l}\gamma_J \simeq 4\Gamma_{cusp}\ln\frac{\mu}{\sqrt{\tau}Q} \\ \gamma_S \simeq -4\Gamma_{cusp}\ln\frac{\mu}{\tau Q}\end{array}\right\} \Rightarrow 2\gamma_J + \gamma_S \simeq 4\Gamma_{cusp}\ln\frac{\mu}{Q}$

  Cancel with γ_H

  Free of lower scale

  $\gamma_H + 2\gamma_J + \gamma_S = 0 \Rightarrow$ RG invariant

★ SCET-II

- Observables:

  Transverse Momentum Dependent (TMD)

  Jet Broadening, Eenergy-Eenergy-Correlation,

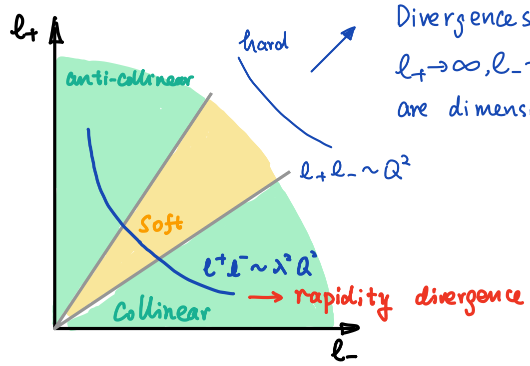
  Massive Form Factor (e.g. EW Sudakov, $h \to b\bar{b}^\gamma \to \gamma\gamma$)

  ... ...

- Scale

	$\ell_+,\ \ell_-,\ \ell_\perp$
Hard:	$Q(1, 1, 1)$
collinear:	$Q(\lambda^2, 1, \lambda)$
Soft:	$Q(\lambda, \lambda, \lambda)$

  Soft and Collinear have the same virtuality!



Divergences from $\ell_+ \to \infty, \ell_- \to \infty$ (UV) are dimensional regulated

$\int_0^\infty \frac{d\ell_-}{\ell_-}$

- Dimensional & rapidity regulatrization

  $\left(\frac{\nu}{\ell_-}\right)^\eta,\ \left|\frac{\ell_+ - \ell_-}{\nu}\right|^\eta,\ e^{\frac{\ell_+ + \ell_-}{\nu}},\ \frac{1}{\ell_- + \Delta},\ \ldots$

- RGE

  Naively, $\gamma_{J,S} \simeq a_{J,S}\Gamma_{cusp}\ln\frac{\mu}{\lambda Q}$

  $2\gamma_J + \gamma_S$ can NOT give hard logarithm

  $\frac{d}{d\ln\mu}[J \otimes J \otimes S] \simeq 4\Gamma_{cusp}\ln\frac{\mu}{Q}$

  Two approaches:

  ▶ Collinear Anomaly — by Becher & Neubert 1007.4005

  ▶ $\boxed{\text{Rapidity Renormalization Group}}$

  — by Chiu, Jain, Neill, Rothstein 1104.0881, 1202.0814

2. Method of Region

Light-cone decomposition:

$$p^\mu = \overset{p^+}{(n\cdot p)}\frac{\bar{n}^\mu}{2} + \overset{p^-}{(\bar{n}\cdot p)}\frac{n^\mu}{2} + p_\perp^\mu = (p_+, p_-, p_\perp).$$

with $\quad n^\mu = (1, \vec{n}), \quad \bar{n}^\mu = (1, -\vec{n}), \quad |\vec{n}| = 1$

For observable $\omega \ll 1$, how to obtain radiative corrections to leading power contribution in ω efficiently?

Factorization

$$\sigma(\omega) = H \otimes \prod_i J_i \otimes S + O(\omega)$$

Factorization can be understood by Method of Region

• Example: thrust in e^+e^- collisions

Definition:

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}, \qquad \tau = 1 - T$$

when $T \to 1$ $(\tau \to 0)$, all the emissions become collinear to $n^\mu / \bar{n}^\mu$, or soft

$$T = \frac{Q - n\cdot\sum P_{Li} - n\sum P_{Ri}}{Q} \quad \Rightarrow \quad \tau = \frac{n\cdot\sum P_{L,i} + \bar{n}\cdot\sum P_{R,i}}{Q}$$

$$\text{collinear: } p_c^\mu \sim Q(\tau, 1, \sqrt{\tau}),$$

$$\text{anti-collinear: } p_{\bar{c}}^\mu \sim Q(1, \tau, \sqrt{\tau}),$$

$$\text{soft: } p_s^\mu \sim Q(\tau, \tau, \tau)$$

$$\left.\begin{array}{l} P_{c,\bar{c}}^2 \sim Q^2\tau \\ P_s^2 \sim Q^2\tau^2 \end{array}\right\} \Rightarrow \text{SCET-1}.$$

Soft momenta can NOT be transformed to be collinear by boost

NLO calculation in QCD

Leptonic tensor : $L_{\mu\nu} = e^2 \sum_{spin} [\bar v(l_2)\gamma_\mu u(l_1)][\bar u(l_1)\gamma_\nu v(l_2)] = \frac{e^2}{4}\left[4\left(-g_{\mu\nu}\, l_1\cdot l_2 + l_{1\mu} l_{2\nu} + l_{2\mu} l_{1\nu}\right)\right]$

$$Q_{\mu\nu} = N_c e_q^2 \sum_{spin} [\bar u(p_1)\gamma_\mu v(p_2)][\bar v(p_2)\gamma_\nu u(p_1)]$$

$s = (p_1+p_2)^2$
$t = (l_1-p_1)^2 = (l_2-p_2)^2$
$u = (l_2-p_1)^2 = (l_1-p_2)^2$

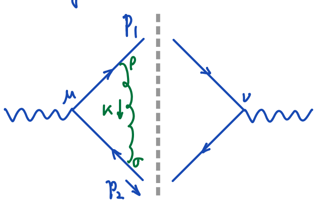
Born Amplitude : $|\mathcal{M}_0|^2 = \frac{1}{s^2} L_{\mu\nu} Q^{\mu\nu}$

$$= N_c \frac{1}{s^2} e^2 e_q^2 \left(-g_{\mu\nu}\frac{s}{2} + l_{1\mu} l_{2\nu} + l_{2\mu} l_{1\nu}\right)\cdot 4\left(-g^{\mu\nu}\frac{s}{2} + p_1^\mu p_2^\nu + p_2^\mu p_1^\nu\right)$$

$$= N_c e^2 e_q^2 \frac{4}{s^2}\left[d\frac{s^2}{4} - \frac{s^2}{4}\cdot 4 + 2(l_1\cdot p_1)(l_2\cdot p_2) + 2(l_1\cdot p_2)(l_2\cdot p_1)\right]$$

$$|\mathcal{M}_0|^2 = N_c e^2 e_q^2 \frac{2}{s^2}\left(t^2+u^2-\varepsilon s^2\right)$$

Diag. A



$p_1^\mu = \frac{\sqrt{s}}{2} n^\mu = \sqrt{s}(0,1,0)$, $p_2^\mu = \frac{\sqrt{s}}{2}\bar n^\mu = \sqrt{s}(1,0,0)$,

$$\mathcal{M}_A \mathcal{M}_0^* = \frac{1}{s^2} L_{\mu\nu}\, N_c C_F e_q^2 (ig_s)^2 \int \frac{d^dk}{(2\pi)^d} \frac{-ig_{\rho\sigma}}{k^2}\frac{i}{(p_1+k)^2}\frac{i}{(p_2-k)^2}$$
$$\times \sum_{spin}[\bar u(p_1)\gamma^\rho(\not p_1+\not k)\gamma^\mu(\not k-\not p_2)\gamma^\sigma v(p_2)][\bar v(p_2)\gamma^\nu u(p_1)]$$

hard : $k^\mu \sim Q(1,1,1)$

$$\int d^dk \frac{(-1)^{a+b+c}}{(k^2)^a[(k+p_1)^2]^b[(k-p_2)^2]^c} = i\pi^{\frac{d}{2}}\frac{\Gamma(a+b+c-\frac{d}{2})}{\Gamma(a)\Gamma(b)\Gamma(c)}\int_0^\infty dx\, x^{a-1}\int_0^\infty dy\, y^{b-1}\int_0^\infty dz\, z^{c-1}\,\delta(1-x-y-z)\,[yz(-2p_1\cdot p_2)]^{\frac{d}{2}-a-b-c}$$

$$= i\pi^{\frac{d}{2}}\frac{\Gamma(a+b+c-\frac{d}{2})}{\Gamma(a)\Gamma(b)\Gamma(c)}\int_0^1 dy\, y^{\frac{d}{2}-a-c-1}\int_0^{1-y} dz\, z^{\frac{d}{2}-a-b-1}(1-y-z)^{a-1}(-s)^{\frac{d}{2}-a-b-c}$$

$$= i\pi^{\frac{d}{2}}\frac{\Gamma(a+b+c-\frac{d}{2})}{\Gamma(a)\Gamma(b)\Gamma(c)}\int_0^1 dy\, y^{\frac{d}{2}-a-c-1}(1-y)^{\frac{d}{2}-b-1}\int_0^1 dz\, z^{\frac{d}{2}-a-b-1}(1-z)^{a-1}(-s)^{\frac{d}{2}-a-b-c}$$

$$\Rightarrow \quad I_h(a,b,c) = (-1)^{a+b+c}\, i\pi^{\frac{d}{2}}\frac{\Gamma(a+b+c-\frac{d}{2})\Gamma(\frac{d}{2}-a-c)\Gamma(\frac{d}{2}-a-b)}{\Gamma(b)\Gamma(c)\Gamma(d-a-b-c)}(-s)^{\frac{d}{2}-a-b-c}$$

using relations:

$$\gamma^\nu\gamma_\nu = d$$
$$\gamma^\nu\gamma^\mu\gamma_\nu = (2-d)\gamma^\mu$$
$$\gamma^\nu\gamma^\mu\gamma^\rho\gamma_\nu = -(4-d)\gamma^\mu\gamma^\rho + 4g^{\mu\rho}$$
$$\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\nu = (4-d)\gamma^\rho\gamma^\mu\gamma^\sigma - 2\gamma^\sigma\gamma^\mu\gamma^\rho$$

$$\gamma^\sigma\gamma^\mu\gamma^\rho = -\gamma^\mu\gamma^\sigma\gamma^\rho + 2g^{\mu\sigma}\gamma^\rho$$
$$= +\gamma^\mu\gamma^\rho\gamma^\sigma - 2g^{\rho\sigma}\gamma^\mu + 2g^{\mu\sigma}\gamma^\rho$$
$$= -\gamma^\rho\gamma^\mu\gamma^\sigma + 2g^{\mu\rho}\gamma^\sigma - 2g^{\rho\sigma}\gamma^\mu + 2g^{\mu\sigma}\gamma^\rho$$

$$\gamma^\rho(\not p_1+\not k)\gamma^\mu(\not k-\not p_2)\gamma_\rho = (k_\rho k_\sigma + p_{1\rho}k_\sigma - k_\rho p_{2\sigma} - p_{1\rho}p_{2\sigma})\left[(4-d)\gamma^\rho\gamma^\mu\gamma^\sigma - 2\gamma^\sigma\gamma^\mu\gamma^\rho\right]$$

$$= (k_\rho k_\sigma + p_{1\rho}k_\sigma - k_\rho p_{2\sigma} - p_{1\rho}p_{2\sigma})\left[(6-d)\gamma^\rho\gamma^\mu\gamma^\sigma - 4g^{\mu\rho}\gamma^\sigma + 4g^{\rho\sigma}\gamma^\mu - 4g^{\mu\sigma}\gamma^\rho\right]$$

the terms survive due to Dirac equation $\bar u(p)\not p$

$$\sim (6-d)\not k\gamma^\mu\not k - 8k^\mu\not k + 4k^2\gamma^\mu - 4p_1\cdot p_2\gamma^\mu - 4p_1^\mu\not k + 4p_1\cdot k\gamma^\mu - 4p_2\cdot k\gamma^\mu + 4p_2^\mu\not k$$

$$= (d-2)k^2\gamma^\mu - 4p_1\cdot p_2\gamma^\mu + 4p_1\cdot k\gamma^\mu - 4p_2\cdot k\gamma^\mu + \underline{(4-2d)k^\mu\not k} - \underline{4p_1^\mu\not k} + \underline{4p_2^\mu\not k}$$

$k^\mu k^\rho$ leads to $\{g^{\mu\rho}k^2,\ p_1^\mu p_2^\rho,\ p_1^\rho p_2^\mu\}$ only the first survive due to Dirac equation

integrals corresponding to $\not k$ give $C_1\not p_1 + C_2\not p_2$, vanish due to Dirac equation

Because of Dirac equation, Considering the loop integration $k^\mu \not{k}$ contributes to amplitude as

$$\bar{u}(p_1) k^\mu \not{k}\, v(p_2) = C(p_1, p_2, k)\, \bar{u}(p_1)\gamma^\mu v(p_2)$$

To determine coefficient $C(p_1, p_2, k)$, we use:

$$\mathrm{Tr}[\not{p}_1 \not{k} \not{p}_2 \not{k}] = C(p_1, p_2, k)\, \mathrm{Tr}[\not{p}_1 \gamma^\mu \not{p}_2 \gamma_\mu]$$

$$\Rightarrow \; -k^2 \mathrm{Tr}(\not{p}_1 \not{p}_2) + 2p_2\cdot k\, \mathrm{Tr}[\not{p}_1 \not{k}] = C(p_1, p_2, k)\,(2-d)\,\mathrm{Tr}(\not{p}_1 \not{p}_2)$$

$$\Rightarrow \; C(p_1, p_2, k) = \frac{1}{2-d}\left[-k^2 + \frac{2(p_1\cdot k)(p_2\cdot k)}{p_1\cdot p_2}\right]$$

Finally, the numerator can be simplified as

$$\Rightarrow \; \bar{u}(p_1)\gamma^\rho(\not{p}_1+\not{k})\gamma^\mu(\not{k}-\not{p}_2)\gamma_\rho v(p_2) \;\to\; \left[(d-4)k^2 - 4p_1\cdot p_2 + 4p_1\cdot k - 4p_2\cdot k + 4\frac{(k\cdot p_1)(k\cdot p_2)}{p_1\cdot p_2}\right]\bar{u}(p_1)\gamma^\mu v(p_2)$$

Now amplitude $\mathcal{M}_a$ in hard region can be written as

$$\mathcal{M}_A\cdot\mathcal{M}_0^* = \frac{1}{s^2} L_{\mu\nu} N_c C_F e_q^2 (ig_s)^2 \int \frac{d^dk}{(2\pi)^d} \frac{-ig_{\rho\sigma}}{k^2} \frac{i}{(p_1+k)^2} \frac{i}{(p_2-k)^2}$$

$$\times \sum_{\text{spin}} [\bar{u}(p_1)\gamma^\rho(\not{p}_1+\not{k})\gamma^\mu(\not{k}-\not{p}_2)\gamma^\sigma v(p_2)][\bar{v}(p_2)\gamma^\nu u(p_1)]$$

$$= \frac{1}{s^2} L_{\mu\nu} Q^{\mu\nu} \frac{-ig_s^2}{(2\pi)^d}\left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^\varepsilon C_F\left[(d-8)I_h(0,1,1) - 2S\, I_h(1,1,1) - \frac{2}{S} I_h(-1,1,1)\right]$$

$$= |\mathcal{M}_0|^2 \frac{g_s^2}{16\pi^2} C_F \left(\frac{\mu^2}{-S}\right)^\varepsilon \left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} + \frac{\pi^2}{6} - 8 + \left(\frac{14\zeta_3}{3} - 16 + \frac{\pi^2}{4}\right)\varepsilon + O(\varepsilon^2)\right]$$

$$= |\mathcal{M}_0|^2 \frac{\alpha_s}{4\pi} C_F \left[-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-3 - 2\ln\frac{\mu^2}{-S}\right) - \ln^2\frac{\mu^2}{-S} - 3\ln\frac{\mu^2}{-S} + \frac{\pi^2}{6} - 8 + O(\varepsilon)\right]$$

$$\Rightarrow \; \sigma_{\text{diag }A} = \delta(\tau)\,\sigma_0\, h^{(1)}$$

$$h^{(1)} = \frac{\alpha_s}{4\pi} C_F \left[-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-3 - 2\ln\frac{\mu^2}{-S}\right) - \ln^2\frac{\mu^2}{-S} - 3\ln\frac{\mu^2}{-S} + \frac{\pi^2}{6} - 8 + O(\varepsilon)\right]$$

collinear: $k^\mu \sim Q(\tau, 1, \sqrt{\tau})$

$$(k+p_1)^2 = \underbrace{k^2}_{Q^2\tau} + \underbrace{2k\cdot p_1}_{Q^2\tau} \to (k+p_1)^2 \qquad (k-p_2)^2 = \underbrace{k^2}_{Q^2\tau} \ll \underbrace{-2k\cdot p_2}_{Q^2} \to -2k\cdot p_2$$

Collinear integral turns to be:

$$\int d^dk \frac{1}{k^2(k+p_1)^2(-2k\cdot p_2)} \sim -(i\pi)^{\frac{d}{2}}\,\Gamma\!\left(3-\tfrac{d}{2}\right)\int_0^1 dx \int_0^\infty dy\, [xy(-s)]^{\frac{d}{2}-3} \leftarrow \text{Scaleless}$$

Soft: $k^\mu \sim Q(\tau, \tau, \tau)$

$$(k+p_1)^2 = \underbrace{k^2}_{Q^2\tau^2} \ll \underbrace{2k\cdot p_1}_{Q^2\tau} \to 2k\cdot p_1, \qquad (k-p_2)^2 = \underbrace{k^2}_{Q^2\tau^2} \ll \underbrace{-2k\cdot p_2}_{Q^2\tau} \to -2k\cdot p_2$$

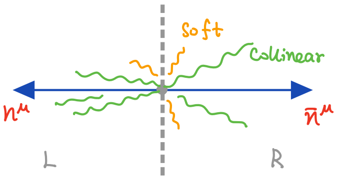
Soft integral turns to be

$$\int d^dk \frac{1}{k^2(2k\cdot p_1)(-2k\cdot p_2)} \quad \begin{cases} (p_1\cdot p_2)^{2d-3}\, I(\varepsilon) & \text{by dimensional analysis} \\ (p_1\cdot p_2)^{-1}\, I(\varepsilon) & \text{by rescale of } p_1 \text{ and } p_2 \end{cases} \Bigg\} \text{ Contradictory if } I(\varepsilon) \neq 0$$

For real radiation

$$\tau = \frac{n\cdot\sum p_{L,i} + \bar{n}\cdot\sum p_{R,i}}{Q} \to 0$$ demands k^μ should be collinear to p_1^μ or p_2^μ, or be soft

$$= \frac{1}{Q^2}\Big(\underbrace{p_L^2 + p_R^2}_{\text{Jet func.}} + \underbrace{Q k_L^+ + Q k_R^-}_{\text{Soft func.}}\Big)$$



Phase space integration

$$\int d\Pi_{q\bar{q}g} = \frac{1}{2s}\int\frac{d^dp_1}{(2\pi)^d}(2\pi)\delta(p_1^2)\int\frac{d^dp_2}{(2\pi)^d}(2\pi)\delta(p_2^2)\int\frac{d^dk}{(2\pi)^d}(2\pi)\delta(k^2)\,(2\pi)^d\delta^{(d)}(q-p_1-p_2-k)$$

For k^μ collinear to p_1^μ

$$\left.\begin{aligned} k_1^\mu &\sim \bar{n}\cdot k\,(\tau, 1, \sqrt{\tau}) \\ p_1^\mu &\sim \bar{n}\cdot p_1\,(\tau, 1, \sqrt{\tau}) \\ p_2^\mu &\sim n\cdot p_2\,(1, \tau, \sqrt{\tau}) \end{aligned}\right\} \Rightarrow \begin{aligned} 2p_1\cdot k &= \underbrace{p_1^+k^-}_{\tau} + \underbrace{p_1^-k^+}_{\tau} + \underbrace{2p_{1\perp}\cdot k_\perp}_{\tau} = (p_1+k)^2 \\ 2p_2\cdot k &= \underbrace{p_2^+k^-}_{1} + \underbrace{p_2^-k^+}_{\tau^2} + \underbrace{2p_{2\perp}\cdot k_\perp}_{\tau} = (n\cdot p_2)\cdot(\bar{n}\cdot k) \simeq Q(\bar{n}\cdot k) \end{aligned}$$

$$\int d\Pi_{q\bar{q}g}(\tau) = \int dp^2\,\delta\!\left(\tau-\frac{p^2}{s}\right)\underbrace{\frac{1}{2s}\int\frac{d^dp_2}{(2\pi)^d}(2\pi)\delta^+(p_2^2)\int\frac{d^dp_c}{(2\pi)^d}(2\pi)\delta^+(p_c^2-p^2)\,(2\pi)^d\delta^{(d)}(q-p_c-p_2)}_{\int d\Pi_2(p_2,\,p)}\int d\Pi_c(p_c\to p_1+k)$$

$$= \int d\Pi_2(p_2, p)\; s\,\underbrace{\int d\Pi_c(p\to p_1+k)}_{\text{collinear part}}$$

with

$$\int d\Pi_2(p_2,p) = \frac{1}{2\pi}\frac{1}{2s}\int\frac{d^dp_2}{(2\pi)^d}(2\pi)\delta^+(p_2^2)\int\frac{d^dp}{(2\pi)^d}(2\pi)\delta^+(p^2-\tau s)\,(2\pi)^d\delta^{(d)}(q-p-p_2)$$

$p^2 = p_+p_-$

$k_+ = \dfrac{p^2 - \frac{p^2}{p_-}k_-}{p_-}$

$$\int d\Pi_c(p\to p_1+k) = \frac{1}{2\pi}\int\frac{d^dp_1}{(2\pi)^d}(2\pi)\delta^+(p_1^2)\int\frac{d^dk}{(2\pi)^d}(2\pi)\delta^+(k^2)\,(2\pi)^d\delta^{(d)}(p-p_1-k)$$

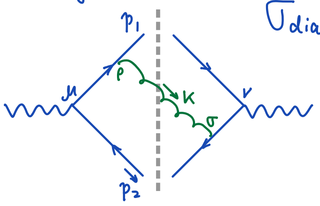
$$= \frac{1}{(2\pi)^{d-1}}\frac{2\pi^{1-\varepsilon}}{\Gamma(1-\varepsilon)}\frac{1}{4}\int dk_+\int dk_-\int dk_T^2\,(k_T^2)^{-\varepsilon}\,\delta(k_+k_- - k_T^2)\,\delta(p^2 - p_+k_- - p_-k_+)$$

$$= \frac{2\pi^{\frac{1}{2}-\varepsilon}}{\Gamma(\frac{1}{2}-\varepsilon)}\int_0^\pi d\phi\,\sin^{-2\varepsilon}\phi$$

$$\Rightarrow \int d\Pi_c(p\to p_1+k) = \frac{e^{\varepsilon\gamma_E}}{(4\pi)^\varepsilon}\frac{1}{\Gamma(1-\varepsilon)}\frac{1}{16\pi^2}(s\tau)^{-\varepsilon}\int_0^{p_-}\frac{dk_-}{p_-}\left(1-\frac{k_-}{p_-}\right)^{-\varepsilon}\left(\frac{k_-}{p_-}\right)^{-\varepsilon}$$

$\overline{\text{MS}}$ factor

Diag. B



$$\sigma_{diagB} = \frac{1}{s^2} L_{\mu\nu} N_c C_F e_q^2 (ig_s)^2 \int d\Pi_2(p_1, p)$$

$$\times \underbrace{\int d\Pi_c(p \to p_1 + k) \sum_{spin} [\bar{u}(p_1)\gamma^\rho(\not p_1+\not k)\gamma^\mu v(p_2)][\bar{v}(p_2)\gamma^\sigma(-\not p_2 - \not k)\gamma^\nu u(p_1)](-g_{\rho\sigma}) \frac{i}{(p_1+k)^2}\frac{i}{(p_2+k)^2}}_{H^{\mu\nu}}$$

For k^μ collinear to p_1^μ

$$L_{\mu\nu}H^{\mu\nu} = \frac{e^2}{4} 16 \left(1 - \frac{\bar{n}\cdot k}{\bar{n}\cdot(p_1+k)}\right) s\,(t^2+u^2-\varepsilon s^2) \frac{i}{p^2}\frac{i}{(n\cdot p_2)(\bar{n}\cdot k)}$$

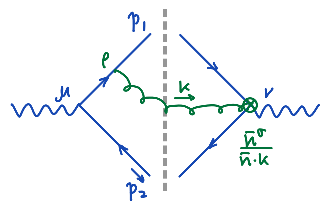
$$|\mathcal{M}_0|^2 = N_c e^2 e_q^2 \frac{2}{s^2}(t^2+u^2-\varepsilon s^2)$$

$$\Rightarrow \sigma^{k\parallel p_1}_{diagB} = \int d\Pi_2(p,p_2)|\mathcal{M}_0|^2 C_F (ig_s)^2 \int d\Pi_c(p\to p_1+k)\, 2\left(1-\frac{\bar{n}\cdot k}{\bar{n}\cdot(p_1+k)}\right)\frac{i}{p^2}\frac{i}{\sqrt{s}\cdot(\bar{n}\cdot k)}$$

$$= \int d\Pi_2(p,p_2)|\mathcal{M}_0|^2 C_F g_s^2 \frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\frac{1}{16\pi^2} s^{1-\varepsilon}\tau^{-\varepsilon}\int_0^{p^-}\frac{dk_-}{p_-}\left(1-\frac{k_-}{p_-}\right)^{-\varepsilon}\left(\frac{k_-}{p_-}\right)^{-\varepsilon} 2\left(1-\frac{k_-}{p_-}\right) s \underbrace{\frac{1}{p^2}}_{=\frac{1}{\tau s}\sim\mathcal{O}(\tau^{-1})}\frac{1}{(n\cdot p_2)k_-}$$

$$\Rightarrow \sigma^{k\parallel p_1}_{diagB} = \underbrace{\frac{C_F\alpha_s}{4\pi}s^{-\varepsilon}\tau^{-1-\varepsilon}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\frac{2\Gamma(2-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(2-2\varepsilon)}}_{s\, j_b(\tau s)}\int d\Pi_2(p,p_2)|\mathcal{M}_0|^2$$

Factorization at diagram level



for k^μ collinear to p_1^μ, using Dirac equation $\not p\, u(p) = \not k\, u(p) = \not p_2 u(p_2) = 0$

$$(ig_s)^2 C_F \sum_{spin}[\bar{u}(p_1)\gamma^\rho(\not p_1+\not k)\gamma^\mu v(p_2)][\bar{v}(p_2)\gamma^\sigma(-\not p_2-\not k)\gamma^\nu u(p_1)](-g_{\rho\sigma})\frac{i}{(p_1+k)^2}\frac{i}{(p_2+k)^2}$$

$p_1 \to p - k$

$$= (ig_s)^2 C_F \sum_{spin}[\bar{v}(p_2)\gamma^\rho(\not p_2+\not k)\gamma^\nu u(p-k)][\bar{u}(p-k)\gamma_\rho \not p \gamma^\mu v(p_2)]\frac{i}{p^2}\frac{i}{2p_2\cdot k}$$

$$= (ig_s)^2 C_F \sum_{spin}[\bar{v}(p_2)\gamma^\rho(\not p_2+\not k)\gamma^\nu(\not p-\not k)\gamma_\rho\not p\gamma^\mu v(p_2)]\frac{i}{p^2}\frac{i}{(n\cdot p_2)(\bar{n}\cdot k)}$$

$$= (ig_s)^2 C_F\left\{\sum_{spin}[\bar{v}(p_2)\gamma^\rho \not p_2\gamma^\nu(\not p-\not k)\gamma_\rho\not p\gamma^\mu v(p_2)] + \underline{\sum_{spin}[\bar{v}(p_2)\gamma^\rho\not k\gamma^\nu(\not p-\not k)\gamma_\rho u(p)][\bar{u}(p)\gamma^\mu v(p_2)]}\right\}$$

$$= \underbrace{(ig_s)^2 C_F}\sum_{spin}[\bar{v}(p_2)\gamma^\nu\underbrace{(\not p-\not k)\gamma_\rho\not p}\gamma^\mu v(p_2)]\underbrace{2p_2^\rho\frac{i}{p^2}\frac{i}{(n\cdot p_2)(\bar{n}\cdot k)}}$$

vanish by using $\not p_2 u(p_2) = \not p\, u(p) = \not k u(p) = 0$ &
$\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\nu = (4-d)\gamma^\rho\gamma^\mu\gamma^\sigma - 2\gamma^\sigma\gamma^\mu\gamma^\rho$

$$\int d\Pi_c(p\to p_2+k)\left[(ig_s)^2 C_F(\not p-\not k)\gamma_\rho\not p\, 2p_2^\rho\frac{i}{p^2}\frac{i}{(n\cdot p_2)(\bar{n}\cdot k)}\right]$$

$$= \int d\Pi_c(p\to p_2+k)\left[C_F g_{\rho\sigma}(\not p-\not k)ig\gamma^\rho\frac{i\not p}{p^2}ig\frac{i\bar{n}^\sigma}{\bar{n}\cdot k}\right]$$

$$= \not p\, j_b(p^2) = \sum_{spin} u(p)\bar{u}(p)\, j_b(p^2)$$

Corresponding to contribution of diagram b in jet func.

$\sigma^{k\parallel p_1}_{diagB}$ can be rewritten as

$$\sigma^{k\parallel p_1}_{diagB} = \frac{1}{s^2}L_{\mu\nu}N_c e_q^2\int d\Pi_2(p_1,p)\sum_{spin}[\bar{u}(p_1)\gamma^\mu v(p_2)][\bar{v}(p_2)\gamma^\nu u(p_1)]\, s\, j_b(p^2)$$

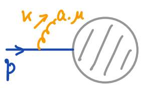
$$= \sigma_0\cdot s\, j_b(\tau s)$$

For soft k^μ

• Eikonal approximation

$$p^\mu = \bar{n}\cdot p \,\frac{n^\mu}{2}, \qquad \bar{n}\cdot p \sim Q, \qquad k^\mu \sim Q(\tau,\tau,\tau)$$

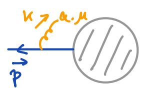
Incoming quark



$$\frac{i\not{p}}{-2p\cdot k+i0} \simeq \frac{i(\not{p}-\not{k})}{(p-k)^2+i0}\, ig_s\gamma^\mu t^a_{ji}\, u(p)$$

$$= u(p)\,(-t^a_{ji})\, ig_s n^\mu \frac{i}{n\cdot k - i0}$$

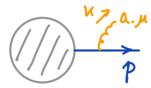
Incoming anti-quark



$$\bar{v}(p)\, ig_s\gamma^\mu t^a_{ij} \frac{i(-\not{p}+\not{k})}{(p-k)^2+i0}$$

$$= \bar{v}(p)\, ig_s n^\mu \frac{i}{n\cdot k - i0}\, t^a_{ij}$$

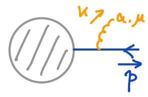
Outgoing quark



$$\bar{u}(p)\, ig_s\gamma^\mu t^a_{ij} \frac{i(\not{p}+\not{k})}{(p+k)^2+i0}$$

$$= \bar{v}(p)\, ig_s n^\mu \frac{i}{n\cdot k + i0}\, t^a_{ij}$$

Outgoing anti-quark



$$\frac{i(-\not{p}-\not{k})}{(p+k)^2+i0}\, ig_s\gamma^\mu t^a_{ji}\, u(p)$$

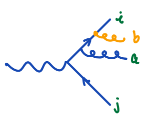
$$= v(p)\, ig_s n^\mu \frac{i}{n\cdot k - i0}\, (-t^a_{ji})$$

So it has "uniform" kinematic part : $ig_s n^\mu \dfrac{i}{n\cdot k \pm i0}$

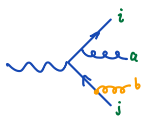
but distinct for color:
$$T^a|c\rangle = \begin{cases} t^a_{ij} & \text{for incoming ant-quark / outgoing quark} \\ -t^a_{ji} & \text{for incoming quark / outgoing anti-quark} \\ if^{abc} & \text{for gluon} \end{cases}$$

color conservation : $\displaystyle\sum_{\text{over ext. legs}} T^a_i|c\rangle = 0$ Catani-Seymour formalism

Example:



$+$



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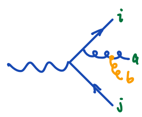


$$t^b_{ik}\,t^a_{kj} + t^a_{ik}(-t^b_{kj}) + if^{abc}t^c_{ij} = -[t^a,t^b]_{ij} + if^{abc}t^c_{ij} = 0$$

$$\Rightarrow \sigma^{\text{soft}}_{\text{diagB}} = \frac{1}{s^2} L_{\mu\nu}\, e_q^2 \int d\Pi_2(p_1,p)\sum_{\text{spin}}[\bar{u}(p_1)\gamma^\mu v(p_2)][\bar{v}(p_2)\gamma^\nu u(p_1)]$$

$$\times \int \frac{d^dk}{(2\pi)^d}(2\pi)\delta^+(k^2)\left[\theta(\bar{n}\cdot k - n\cdot k)\,\delta\!\left(\tau - \frac{n\cdot k}{\sqrt{s}}\right) + \theta(n\cdot k - \bar{n}\cdot k)\,\delta\!\left(\tau - \frac{\bar{n}\cdot k}{\sqrt{s}}\right)\right]$$

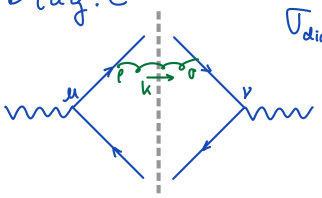
$$\times \underbrace{\text{tr}[t^a(-t^a)]}_{-N_c C_F}\; ig_s n^\rho \frac{i}{n\cdot k+i0}\left(ig_s\bar{n}^\rho \frac{i}{\bar{n}\cdot k+i0}\right)^*(-g_{\rho\sigma})$$

$$= \sigma_0\, C_F\, g_s^2\, \frac{e^{\varepsilon\gamma_E}}{(4\pi)^\varepsilon}\frac{2\pi^{1-\varepsilon}}{\Gamma(1-\varepsilon)}\frac{1}{(2\pi)^{3-2\varepsilon}}$$

$$\frac{1}{4}\left[\int_{k_+}^\infty dk_-\int_0^\infty dk_+\,\delta\!\left(\tau - \frac{k_+}{\sqrt{s}}\right) + \int_{k_-}^\infty dk_+\int_0^\infty dk_-\,\delta\!\left(\tau-\frac{k_-}{\sqrt{s}}\right)\right](k_+k_-)^{-\varepsilon}\frac{2}{k_+k_-}$$

$$\Rightarrow \boxed{\sigma^{\text{soft}}_{\text{diagB}} = \sigma_0\,\frac{C_F\alpha_s}{4\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\frac{4}{\varepsilon}\,s^{-\varepsilon}\,\tau^{-1-2\varepsilon}}$$

Diag. C



$$\sigma_{diagC} = \frac{1}{s^2} L_{\mu\nu} N_c C_F e_q^2 (ig_s)^2 \int d\Pi_2(p_1, p)$$

$$\times \int d\Pi_c(p \to p_2 + k) \underbrace{\sum_{spin} \left[\bar{u}(p_1)\gamma^\rho(\not{p}_1 + \not{k})\gamma^\mu v(p_2)\right]\left[\bar{v}(p_2)\gamma^\nu(\not{p}_1 + \not{k})\gamma^\sigma u(p_1)\right](-g_{\rho\sigma})\left[\frac{i}{(p_1+k)^2}\right]^2}_{H^{\mu\nu}}$$

$$n\cdot \ell_1 = \bar{n}\cdot \ell_2 = -\frac{t}{\sqrt{s}}$$

$$\bar{n}\cdot \ell_1 = n\cdot \ell_2 = -\frac{u}{\sqrt{s}}$$

$$n\cdot p = \bar{n}\cdot p_1 = \sqrt{s}$$

For k^μ collinear to p_2^μ

$$L_{\mu\nu} H^{\mu\nu} = 8(\varepsilon - 1)\left[\underbrace{s(k\cdot p_2)}_{O(Q^4\tau)} - 2(\underbrace{k\cdot \ell_2\ p_2\cdot \ell_1}_{O(Q^4)} + \underbrace{k\cdot \ell_1\ p_2\cdot \ell_2}_{O(Q^4)})\right] \frac{-1}{(n\cdot k)(\bar{n}\cdot p_1)}$$

$$\sigma^{k/\!/p_2}_{diag\,C} = \frac{1}{s^2} N_c e^2 e_q^2 \int d\Pi_2(p_1, p) \int d\Pi_c(p \to p_2 + k) \underbrace{\frac{4(1-\varepsilon)}{(n\cdot k)(\bar{n}\cdot p_1)} \left[(n\cdot k)(\bar{n}\cdot \ell_2)(n\cdot p_2)(\bar{n}\cdot \ell_1) + (n\cdot k)(\bar{n}\cdot \ell_1)(n\cdot p_2)(\bar{n}\cdot \ell_2)\right]}_{O(\tau^0) \text{ power suppressed comparing to } O(\tau^{-1})}$$

For k^μ collinear to p_1^μ

$$(\not{p}_1 + \not{k})\gamma_\rho \not{p}_1 \gamma^\rho (\not{p}_1 + \not{k}) = -\not{k}\not{p}_1\gamma_\rho\gamma^\rho(\not{p}_1 + \not{k}) + 2(\not{p}_1 + \not{k})\not{p}_1(\not{p}_1 + \not{k}) = (-d+2)\not{k}\not{p}_1\not{k} = (-d+2)\, 2k\cdot p_1 \not{k}$$

$$\Rightarrow \sum_{spin} \left[\bar{u}(p_1)\gamma^\rho(\not{p}_1 + \not{k})\gamma^\mu v(p_2)\right]\left[\bar{v}(p_2)\gamma^\nu(\not{p}_1 + \not{k})\gamma^\sigma u(p_1)\right](-g_{\rho\sigma})$$

$$= 2(1-\varepsilon)\frac{(\bar{n}\cdot k)\, 2k\cdot p_1}{\bar{n}(k+p_1)} \sum_{spin} \left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]\left[\bar{u}(p_1)\gamma^\mu v(p_2)\right]$$

$$\gamma^\rho\gamma_\rho = d$$

$$\not{k} = \bar{n}\cdot k \frac{\not{n}}{2} = \frac{\bar{n}\cdot k}{\bar{n}\cdot p}\not{p}$$

$$\Rightarrow \sigma^{k/\!/p_1}_{diagC} = \frac{1}{s^2} L_{\mu\nu} N_c e_q^2 \int d\Pi_2(p_1, p) \sum_{spin} \left[\bar{v}(p_2)\gamma^\nu u(p_1)\right]\left[\bar{u}(p_1)\gamma^\mu v(p_2)\right]$$

$$\times\ s \int d\Pi_c(p \to p_2 + k)\ C_F (ig_s)^2\, 2(1-\varepsilon)\frac{(\bar{n}\cdot k)\, 2k\cdot p_1}{\bar{n}\cdot p}\left(\frac{i}{p^2}\right)^2$$

$$= \int d\Pi_2(p_1, p)\, |\mathcal{M}_0|^2 \cdot s \cdot C_F g_s^2 \frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)} \frac{1}{16\pi^2}(\tau s)^{-\varepsilon} \int_0^{p^-} \frac{dk_-}{p_-}\left(1 - \frac{k_-}{p_-}\right)^{-\varepsilon}\left(\frac{k_-}{p_-}\right)^{-\varepsilon} 2(1-\varepsilon)\frac{k_-}{p_-}\frac{1}{p^2} \qquad \text{with } \tau s = p^2$$

$$\Rightarrow \sigma^{k/\!/p_1}_{diagC} = \frac{C_F\alpha_s}{4\pi} s^{-\varepsilon}\tau^{-1-\varepsilon}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\frac{2\Gamma^2(2-\varepsilon)}{\Gamma(3-2\varepsilon)} \int d\Pi_2(p_1, p)\, |\mathcal{M}_0|^2$$

Factorization of diagram

$$\int d\Pi_c(p \to p_2 + k)\, C_F (ig_s)^2 \left[(\not{p}_1 + \not{k})\gamma^\sigma \not{p}_1 \gamma^\rho(\not{p}_1 + \not{k})\right](-g_{\rho\sigma})\left[\frac{i}{(p_1+k)^2}\right]^2$$

$$= \int d\Pi_c(p \to p_2 + k)\, C_F\left[\frac{i\not{p}}{p^2}\, ig\gamma^\sigma(\not{p} - \not{k})\, ig\gamma^\rho \frac{i\not{p}}{p^2}\right](-g_{\rho\sigma})$$

$$= \not{p}\, j_a(p^2)$$

Inserting it back to integration: $\sigma^{k/\!/p_1}_{diagC} = \sigma_0\, s\, j_a(\tau s)$

The soft contribution of Diag. C is 0.

In the following. we use "+" function to express the expansion of $\tau^{-1+\eta}$

$$\tau^{-1+\eta} = \frac{1}{\eta}\delta(\tau) + \left(\frac{1}{\tau}\right)_+ + \eta\left(\frac{\ln\tau}{\tau}\right)_+ + O(\eta^2)$$

$$\int_0^a d\tau\, f(\tau)\,[g(\tau)]_+ = \int_0^a d\tau\,[f(\tau)-f(0)]\,g(\tau) + f(0)\int_1^a d\tau\, g(\tau)$$

To show this,

$$\int_0^a d\tau\, \underline{f(\tau)}\,\tau^{-1+\eta} = \frac{1}{\eta}\int_0^a d\tau^\eta\, f(\tau)$$

convegent at $\tau=0$

$$= \frac{1}{\eta}\tau^\eta f(\tau)\Big|_0^a - \lim_{\delta\to 0}\frac{1}{\eta}\int_\delta^a df(\tau)\,\tau^\eta$$

$$= \frac{1}{\eta}a^\eta f(a) - \lim_{\delta\to 0}\frac{1}{\eta}\int_\delta^a df(\tau)\left(1+\sum_{n=1}^\infty\frac{\eta^n}{n!}\ln^n\tau\right)$$

$$= \frac{1}{\eta}a^\eta f(a) - \lim_{\delta\to 0}\frac{1}{\eta}f(\tau)\left(1+\sum_{n=1}^\infty\frac{\eta^n}{n!}\ln^n\tau\right)\Big|_\delta^a + \lim_{\delta\to 0}\frac{1}{\eta}\int_\delta^a d\tau\, f(\tau)\sum_{n=1}^\infty\frac{\eta^n}{(n-1)!}\frac{\ln^{n-1}\tau}{\tau}$$

$$= \frac{1}{\eta}f(0) - \lim_{\delta\to 0}\sum_{n=1}^\infty\frac{\eta^{n-1}}{n!}f(\delta)\left(\ln^n 1-\ln^n\delta\right) + \lim_{\delta\to 0}\int_\delta^a d\tau\, f(\tau)\sum_{n=1}^\infty\frac{\eta^{n-1}}{(n-1)!}\frac{\ln^{n-1}\tau}{\tau}$$

$$= \frac{1}{\eta}f(0) - \lim_{\delta\to 0}f(\delta)\int_\delta^1 d\tau\,\frac{\eta^{n-1}}{(n-1)!}\frac{\ln^{n-1}\tau}{\tau} + \lim_{\delta\to 0}\int_\delta^a d\tau\, f(\tau)\sum_{n=1}^\infty\frac{\eta^{n-1}}{(n-1)!}\frac{\ln^{n-1}\tau}{\tau}$$

$$= \int_0^a d\tau\, f(\tau)\frac{1}{\eta}\delta(\eta) + \sum_{n=0}^\infty\frac{\eta^n}{n!}\left[\int_0^a d\tau\,[f(\tau)-f(0)]\frac{\ln^n\tau}{\tau} + f(0)\int_1^a d\tau\,\frac{\ln^n\tau}{\tau}\right]$$

The series in integration formula defines the expansion of $\tau^{-1+\eta}$

The contribution of virtual correction: (hard)

$$\sigma^{hard} = \sigma_{diag\,A} + \sigma^{*}_{diag\,A}$$
$$= \sigma_0 \cdot \delta(\tau) \frac{C_F \alpha_s}{4\pi} \mathrm{Re}\left[-\frac{4}{\varepsilon^2} + \frac{2}{\varepsilon}\left(-3 - 2\ln\frac{\mu^2}{-s}\right) - 2\ln^2\frac{\mu^2}{-s} - 6\ln\frac{\mu^2}{-s} + \frac{\pi^2}{3} - 16 + o(\varepsilon) \right]$$
$$= \sigma_0 \cdot \delta(\tau) \frac{C_F \alpha_s}{4\pi} \left[-\frac{4}{\varepsilon^2} + \frac{2}{\varepsilon}\left(-3 - 2\ln\frac{\mu^2}{s}\right) - 2\ln^2\frac{\mu^2}{s} - 6\ln\frac{\mu^2}{s} + \frac{7\pi^2}{3} - 16 + o(\varepsilon) \right]$$

The contribution for n-collinear is

$$\sigma^{col.} = \underset{\text{mirror diagram}}{2\sigma^{k//p_1}_{diag\,B}} + \sigma^{k//p_1}_{diag\,C} + \underbrace{\sigma^{k//p_2}_{diag\,C}}_{=0} = \sigma_0\, s\left(2 j_b(\tau s) + j_a(\tau s)\right)$$

flipping diagram

$$\Rightarrow \sigma^{col.} = \sigma_0 \frac{C_F \alpha_s}{4\pi} s^{-\varepsilon} \tau^{-1-\varepsilon} \frac{e^{\varepsilon \gamma_E}}{\Gamma(1-\varepsilon)} \left[2 \times \frac{2\Gamma(2-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(2-2\varepsilon)} + \frac{2\Gamma^2(2-\varepsilon)}{\Gamma(3-2\varepsilon)} \right]$$
$$= \sigma_0 \frac{C_F \alpha_s}{4\pi} \left\{ \frac{4}{\varepsilon^2}\delta(\tau) + \frac{1}{\varepsilon}\left[\left(3 + 4\ln\frac{\mu^2}{s}\right)\delta(\tau) - 4\left(\frac{1}{\tau}\right)_+ \right] \right.$$
$$\left. + 4\left(\frac{\ln\tau}{\tau}\right)_+ + \left(\frac{1}{\tau}\right)_+\left(-4\ln\frac{\mu^2}{s} - 3\right) + \left(2\ln^2\frac{\mu^2}{s} + 3\ln\frac{\mu^2}{s} + 7 - \pi^2\right)\delta(\tau) \right\}$$

$$\sigma^{anti\text{-}col.} = \sigma^{col.}$$

The contribution of soft emission is

$$\sigma^{soft} = 2\sigma^{soft}_{diag\,B} = \sigma_0 \frac{C_F \alpha_s}{4\pi} \frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)} \frac{8}{\varepsilon} \left(\frac{\mu^2}{s}\right)^{\varepsilon} \tau^{-1-2\varepsilon}$$
$$= \sigma_0 \frac{C_F \alpha_s}{4\pi} \left[\frac{8}{\varepsilon} + 8\ln\frac{\mu^2}{s} + \left(4\ln^2\frac{\mu^2}{s} - \frac{2\pi^2}{3}\right)\varepsilon + o(\varepsilon^2) \right] \left[-\frac{1}{2\varepsilon}\delta(\tau) + \left(\frac{1}{\tau}\right)_+ - 2\varepsilon\left(\frac{\ln\tau}{\tau}\right)_+ + o(\varepsilon^2) \right]$$
$$= \sigma_0 \frac{C_F \alpha_s}{4\pi} \left\{ -\frac{4}{\varepsilon^2}\delta(\tau) + \frac{1}{\varepsilon}\left[8\left(\frac{1}{\tau}\right)_+ - 4\,\delta(\tau)\ln\frac{\mu^2}{s} \right] - 16\left(\frac{\ln\tau}{\tau}\right)_+ + 8\left(\frac{1}{\tau}\right)_+ \ln\frac{\mu^2}{s} + \left(\frac{\pi^2}{3} - 2\ln^2\frac{\mu^2}{s}\right)\delta(\tau) \right\}$$

Eventually, we can obtain NLO thrust distribution by adding up all the contribution:

$$\frac{d\sigma^{NLO}}{d\tau} = \sigma^{hard} + \sigma^{col.} + \sigma^{anti\text{-}col} + \sigma^{soft}$$
$$= \sigma_0 \frac{C_F \alpha_s}{4\pi} \left[-8\left(\frac{\ln\tau}{\tau}\right)_+ - 6\left(\frac{1}{\tau}\right)_+ + \delta(\tau)\left(\frac{2\pi^2}{3} - 2\right) \right]$$

$$\Rightarrow \int_0^{\tau_{cut}} d\sigma^{NLO} = \sigma_0 \frac{C_F \alpha_s}{4\pi} \left(-4\ln^2\tau_{cut} - 6\ln\tau_{cut} + \frac{2\pi^2}{3} - 2 \right)$$

$$\int_0^{\tau_{cut}} d\tau \left(\frac{\ln^n\tau}{\tau}\right)_+$$
$$= -\int_{\tau_{cut}}^{1} d\tau \frac{\ln^n\tau}{\tau}$$
$$= -\frac{1}{n+1}\ln^{n+1}\tau \Big|_{\tau_{cut}}^{1}$$
$$= \frac{1}{n+1}\ln^{n+1}\tau_{cut}$$

3. Fields and Operators in SCET

- Power Counting

quark field Collinear to n^μ

$$\psi_c = \xi + \eta \,, \qquad \text{with} \quad \xi = \underbrace{\frac{\not{n}\not{\bar{n}}}{4}}_{P_+}\psi_c \quad \text{and} \quad \eta = \underbrace{\frac{\not{\bar{n}}\not{n}}{4}}_{P_-}\psi_c \qquad P_+ + P_- = 1$$

ξ: Large component; η: Small component

To determine the power of fields ξ and η, we first consider the collinear propagator:

$$\langle 0|T\{\xi(x)\bar{\xi}(0)\}|0\rangle = \frac{\not{n}\not{\bar{n}}}{4}\langle 0|T\{\psi_c(x)\bar{\psi}_c(0)\}|0\rangle\frac{\not{\bar{n}}\not{n}}{4}$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i0}\frac{\not{n}\not{\bar{n}}}{4}\not{p}\frac{\not{\bar{n}}\not{n}}{4}e^{-ix\cdot p}$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{i(\bar{n}\cdot p)}{p^2+i0}\frac{\not{n}}{2}e^{-ix\cdot p}$$

for collinear momentum p^μ, $\quad p_c^\mu \sim Q\,(\lambda^2, 1, \lambda)$

$$\int d^4p = \frac{1}{4}\int \underbrace{dp_+}_{\lambda^2}\int \underbrace{dp_-}_{1}\int \underbrace{dp_T^2}_{\lambda^2}\int d\phi \quad \Rightarrow \quad \langle 0|T\{\xi(x)\bar{\xi}(0)\}|0\rangle \sim \lambda^4\cdot\frac{1}{\lambda^2} = \lambda^2 \quad \Rightarrow \quad \xi \sim \lambda$$

$$\langle 0|T\{\eta(x)\bar{\eta}(0)\}|0\rangle = \frac{\not{\bar{n}}\not{n}}{4}\langle 0|T\{\psi_c(x)\bar{\psi}_c(0)\}|0\rangle\frac{\not{n}\not{\bar{n}}}{4} = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i0}\frac{\not{\bar{n}}\not{n}}{4}\not{p}\frac{\not{n}\not{\bar{n}}}{4}e^{-ix\cdot p}$$

$$= \int\frac{d^4p}{(2\pi)^4}\frac{i(n\cdot p)}{p^2+i0}\frac{\not{\bar{n}}}{2}e^{-ix\cdot p}$$

$$\sim \lambda^4\cdot\frac{\lambda^2}{\lambda^2} = \lambda^4 \quad \Rightarrow \quad \eta \sim \lambda^2$$

(Ultra-) Soft quark fields

$p_s^\mu \sim Q\,(\lambda^2, \lambda^2, \lambda^2)$

$$\langle 0|T\{\psi_s(x)\bar{\psi}_s(0)\}|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i\not{p}}{p^2+i0}e^{-ip\cdot x} \sim \lambda^8\frac{\lambda^2}{\lambda^4} = \lambda^6$$

$\Rightarrow \quad \psi_s \sim \lambda^3$

Gluon field $A^\mu(x)$

$$\langle 0|T\{A^\mu(x)A^\nu(0)\}|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i0}\left(-g^{\mu\nu}+(1-\xi)\frac{p^\mu p^\nu}{p^2}\right)e^{-ip\cdot x} \sim \int d^4p\,\frac{p^\mu p^\nu}{(p^2)^2} \sim p^\mu p^\nu$$

So A^μ scales as p^μ

$$\Rightarrow \begin{cases} \text{Collinear fields} : & n\cdot A_c \sim Q\lambda^2, \quad \bar{n}\cdot A_c \sim Q, \quad A_{c\perp}^\mu \sim Q\lambda \\ \text{(ultra-) soft fields}: & n\cdot A_s \sim Q\lambda^2, \quad \bar{n}\cdot A_s \sim Q\lambda^2, \quad A_{s\perp}^\mu \sim Q\lambda^2 \end{cases}$$

• Effective Lagrangian

Covariant derivate:

$$iD_\mu = i\partial_\mu + g_s A^a_\mu T^a = i\partial_\mu + g_s\left(A^a_{c,\mu} + A^a_{s,\mu}\right)T^a$$

We first focus on collinear Lagrangian

$$\mathcal{L}_c = \bar\Psi_c\, i\not{D}\,\Psi_c$$

$$= (\bar\xi + \bar\eta)\left[\frac{\not{n}}{2} i\bar n\cdot D + \frac{\not{\bar n}}{2} i n\cdot D + i\not{D}_\perp\right](\xi + \eta)$$

$$= \bar\xi \frac{\not{\bar n}}{2} in\cdot D\, \xi + \bar\eta \frac{\not{n}}{2} i\bar n\cdot D\, \eta + \bar\xi\, i\not{D}_\perp \eta + \bar\eta\, i\not{D}_\perp \xi$$

The equation of motion

$$\partial_\mu \frac{\partial \mathcal{L}_c}{\partial(\partial_\mu \bar\eta)} - \frac{\partial \mathcal{L}_c}{\partial \bar\eta} = 0 \quad\Rightarrow\quad 0 - \left(\frac{\not{n}}{2} i\bar n\cdot D\,\eta + i\not{D}_\perp \xi\right) = 0 \quad\Rightarrow\quad \eta = -\frac{\not{n}}{2\bar n\cdot D}\not{D}_\perp \xi$$

Then it is easy to derive $\bar\eta = -\bar\xi\, \overleftarrow{\not{D}}_\perp \dfrac{\not{n}}{2\bar n\cdot \overleftarrow{D}}$

Insert above two relations into Lagrangian $\mathcal{L}_c$, we can eliminate field η

$$\mathcal{L}_c = \bar\xi \frac{\not{\bar n}}{2} in\cdot D\,\xi + \underline{\bar\xi\, \overleftarrow{\not{D}}_\perp \frac{\not{n}}{2\bar n\cdot\overleftarrow{D}} \frac{\not{n}}{2} i\cancel{\bar n\cdot D} \frac{\not{n}}{2\cancel{\bar n\cdot D}} \not{D}_\perp \xi} - \bar\xi\, i\not{D}_\perp \frac{\not{n}}{2\bar n\cdot D}\not{D}_\perp\xi - \underline{\bar\xi\,\overleftarrow{\not{D}}_\perp \frac{\not{n}}{2\bar n\cdot\overleftarrow{D}} i\not{D}_\perp \xi}$$

$= \bar\xi\,\overleftarrow{\not{D}}_\perp \dfrac{\not{n}}{2\bar n\cdot\overleftarrow{D}} i\not{D}_\perp \xi$ Cancel with

$$\Rightarrow \mathcal{L}_c = \bar\xi \frac{\not{\bar n}}{2} in\cdot D\,\xi + \bar\xi\, i\not{D}_\perp \frac{1}{i\bar n\cdot D} i\not{D}_\perp \frac{\not{n}}{2}\xi$$

$$= \underbrace{\underset{\lambda}{\bar\xi} \frac{\not{\bar n}}{2} \underset{\lambda^2}{in\cdot D}\, \underset{\lambda}{\xi}}_{\lambda^4} + \underbrace{\underset{\lambda}{\bar\xi}\, \underset{\lambda}{i\not{D}_{c\perp}} \underset{1}{\frac{1}{i\bar n\cdot D_c}} \underset{\lambda}{i\not{D}_{c\perp}} \frac{\not{n}}{2}\underset{\lambda}{\xi}}_{\lambda^4} + O(\lambda)$$

← Expansion to subleading power, See M. Beneke et al. 0206152

where $in\cdot D = \underset{\lambda^2}{in\cdot\partial} + g_s \underset{\lambda^2}{n\cdot A_c} + g_s \underset{\lambda^2}{n\cdot A_s}$

for distance conjugate to collinear momentum:

$$x^\mu = (x_+, x_-, x_\perp) \sim (p_{c+}^{-1}, p_{c-}^{-1}, p_{c\perp}^{-1}) \sim (\lambda^{-2}, 1, \lambda^{-1}) \quad\Rightarrow\quad \int d^4x\, \mathcal{L}_c(x) \sim \lambda^{-4}\cdot\lambda^4 \sim 1$$

for distance conjugate to (ultra-)soft momentum:

$$x^\mu \sim (p_{s+}^{-1}, p_{s-}^{-1}, p_{s\perp}^{-1}) \sim (\lambda^{-2}, \lambda^{-2}, \lambda^{-2}) \quad\Rightarrow\quad \int d^4x\, \mathcal{L}_s = \int \underset{\lambda^{-8}}{d^4x}\, \underset{\lambda^3}{\bar\psi_s}\, \underset{\lambda^2}{i\not{D}_s}\, \underset{\lambda^3}{\psi_s} \sim 1$$

Finally, SCET Lagrangian at leading power can be written as

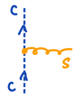
$$\mathcal{L}_{SCET} = \bar\xi \frac{\not{\bar n}}{2} in\cdot D\,\xi + \bar\xi\, i\not{D}_{c\perp} \frac{1}{i\bar n\cdot D_c} i\not{D}_{c\perp}\frac{\not{n}}{2}\xi + \bar\psi_s\, i\not{D}_s\, \psi_s - \frac{1}{4}\left(F^{c,a}_{\mu\nu}\right)^2 - \frac{1}{4}\left(F^{s,a}_{\mu\nu}\right)$$

↳ non-local. can be made explicit in term of Wilson Line.

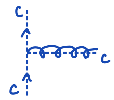
Because both soft and collinear gluon fields are involved in D_μ,

$\bar{\xi}\frac{\not{\bar{n}}}{2}in\cdot D\xi$ gives interactions between collinear quarks and soft / collinear gluon.

$$g_s\, n\cdot A_s^a t^a_{ij}\, \bar{\xi}\frac{\not{\bar{n}}}{2}\xi$$



$$g_s\, n\cdot A_c^a t^a_{ij}\, \bar{\xi}\frac{\not{\bar{n}}}{2}\xi$$



How to remove interaction between collinear fields and soft gluon?

Decouple transformation !

- Multi-pole expansion

We have to expand the position arguments of (ultra-) soft fields in the transverse direction and the direction of $\bar{n}$, since the (ultra-) soft fields vary more slowly than collinear fields in these direction. Defining $x_-^\mu = (\bar{n}\cdot x)\, n^\mu/2$, the relevant expansion is:

$$x^\mu = \underset{1}{(n\cdot x)\frac{\bar{n}^\mu}{2}} + \underset{\lambda^{-2}}{(\bar{n}\cdot x)\frac{n^\mu}{2}} + \underset{\lambda^{-1}}{x_\perp^\mu}$$

$$2x\cdot p_c = \underbrace{\underset{\lambda^{-2}}{(\bar{n}\cdot x)}\underset{\lambda^{2}}{(n\cdot p_c)}}_{O(1)} + \underbrace{\underset{1}{(n\cdot x)}\underset{1}{(\bar{n}\cdot p_c)}}_{O(1)} + \underbrace{\underset{\lambda^{-1}}{x_\perp}\cdot\underset{\lambda}{p_{c\perp}}}_{O(1)}$$

$$2x\cdot p_s = \underbrace{\underset{\lambda^{-2}}{(\bar{n}\cdot x)}\underset{\lambda^{2}}{(n\cdot p_s)}}_{O(1)} + \underbrace{\underset{1}{(n\cdot x)}\underset{\lambda^{2}}{(\bar{n}\cdot p_s)}}_{O(\lambda^2)} + \underbrace{\underset{\lambda^{-1}}{x_\perp}\cdot\underset{\lambda^{2}}{p_{s\perp}}}_{O(\lambda)}$$

$$\begin{aligned}\phi_s(x) &= \phi_s\Big(x_-\frac{n^\mu}{2} + x_+\frac{\bar{n}^\mu}{2} + x_\perp^\mu\Big)\\ &= \phi_s(x_-) + \underbrace{[x_\perp\cdot\partial\phi_s]}_{\lambda}(x_-) + \underbrace{[x_+\frac{\bar{n}}{2}\cdot\partial\phi_s]}_{\lambda^2}(x_-) + \underbrace{[\tfrac{1}{2}x_\perp^\mu x_\perp^\nu\partial_\mu\partial_\nu\phi_s]}_{\lambda^2}(x_-) + O(\lambda^3\phi_s)\end{aligned}$$

When soft fields are involved in interaction with collinear fields.

the collinear distance $x^\mu\sim(1,\lambda^{-2},\lambda^{-1})$ hit soft momentus inhomogenously.

So multi-pole expansion is necessary for power expansion of Lagrangian.

$$\bar{\xi}\frac{\not{\bar{n}}}{2}in\cdot D\xi \supseteq \bar{\xi}(x)\frac{\not{\bar{n}}}{2}n\cdot A_s(x)\xi(x) \xrightarrow[\text{expansion}]{\text{Multi-pole}} \bar{\xi}(x)\frac{\not{\bar{n}}}{2}n\cdot A_s(x_-)\xi(x) + O(\lambda^5)$$

For the pure (ultra-) soft Lagrangian, we do NOT need multi-pole expansion, because $x^\mu\sim(\lambda^{-2},\lambda^{-2},\lambda^{-2})$

For n-$\bar{n}$ current $\int d^4x\,\bar{\xi}_n(x)S_n^\dagger(x)\gamma_\perp^\mu S_{\bar{n}}(x)\xi_{\bar{n}}(x)$, $x^\mu\sim(1,1,1)$,

so No multi-pole expansion.

- Wilson Lines

Collinear: $W_c(x) = \mathbb{P}\exp\left[ig_s\int_{-\infty}^{0}dt\,\bar{n}\cdot A_c(x+t\bar{n})\right]$

(ultra-) soft: $S_n(x) = \mathbb{P}\exp\left[ig_s\int_{-\infty}^{0}dt\,n\cdot A_s(x+tn)\right]$ ← decouple from n-collinear field

An important property is that the covariant derivative alone the Wilson line is zero

$$in\cdot D_s\,S_n(x) = 0 \quad \text{with} \quad D_s^\mu = \partial^\mu - ig_s A_s^\mu(x)$$

Proof: $n\cdot\partial\, S_n(x) = \lim_{\delta\to0}\frac{1}{\delta}\left[S_n(x+\delta n) - S_n(x)\right]$

$$= \lim_{\delta\to0}\frac{1}{\delta}\left\{\mathbb{P}\exp\left[ig_s\int_{-\infty}^{0}dt\,n\cdot A_s(x+(\delta+t)n)\right] - \mathbb{P}\exp\left[ig_s\int_{-\infty}^{0}dt\,n\cdot A_s(x+tn)\right]\right\}$$

$$= \lim_{\delta\to0}\frac{1}{\delta}\left\{\mathbb{P}\exp\left[ig_s\int_{-\infty}^{\delta}dt\,n\cdot A_s(x+tn)\right] - \mathbb{P}\exp\left[ig_s\int_{-\infty}^{0}dt\,n\cdot A_s(x+tn)\right]\right\}$$

$$= \frac{d}{d\tau}\mathbb{P}\exp\left[ig_s\int_{-\infty}^{\tau}dt\,n\cdot A_s(x+tn)\right]\Big|_{\tau=0}$$

$$= ig_s\,n\cdot A_s(x)\,S_n(x)$$

$$\Rightarrow n\cdot\left[\partial - ig_s A_s(x)\right]\cdot S_n(x) = 0$$

$$\Rightarrow in\cdot D\,S_n(x) = 0$$

- Gauge Transformations and Reparameterization Invariance

Soft gauge transformation: $V_s(x) = \exp\left[i\alpha_s^a(x)\,t^a\right]$

$$\psi_s(x) \to V_s(x)\,\psi_s(x), \qquad A_s^\mu(x) \to V_s(x)A_s^\mu V_s^\dagger(x) + \frac{i}{g_s}V_s(x)\left[\partial^\mu, V_s^\dagger(x)\right]$$

$$\xi(x) \to V_s(x_-)\,\xi(x), \qquad A_c^\mu(x) \to V_s(x)A_c^\mu V_s^\dagger(x)$$

Collinear gauge transformation: $V_c(x) = \exp\left[i\alpha_c^a(x)\,t^a\right]$

$$\psi_s(x) \to \psi_s(x), \qquad A_s^\mu(x) \to A_s^\mu(x)$$

$$\xi(x) \to V_c(x)\xi(x), \qquad A_c^\mu(x) \to V_c(x)A_c^\mu(x)V_c^\dagger(x) + \frac{1}{g}V_c(x)\left[i\partial^\mu + g_s n\cdot A_s(x_-)\frac{\bar{n}^\mu}{2}, V_c^\dagger(x)\right]$$

$$W_c(x) \to V_c(x)\,W_c(x)\,V_c^\dagger(-\infty\bar{n})$$

the last transformation ensures that

$$in\cdot D \to V_c\, in\cdot D\, V_c^\dagger$$

Gauge invariant building blocks

$$\chi_n(x) \equiv W_n^\dagger(x)\,\psi_n(x), \qquad \bar{\chi}_n(x) \equiv \bar{\psi}_n(x)\,W_n(x)$$

$$\mathcal{A}_n^\mu(x) \equiv W_n^\dagger(x)\left[iD_n^\mu W_n(x)\right]$$

- **Decoupling Transformation**

As seen in the SCET Lagrangian, $\bar{\xi} \frac{\not{\bar{n}}}{2} i n\cdot D \xi$ involves interaction between collinear and soft fields.

To decouple soft fields from collinear fields, we redefined collinear fields:

$$\xi(x) \rightarrow S_n(x_-)\, \xi^{(0)}(x) \,, \qquad A_c^{\mu}(x) \rightarrow S_n(x_-)\, A_c^{(0)\mu}(x)\, S_n^{+}(x_-)$$

As a consequence of the field transformations, we find:

$$i n\cdot D\, \xi(x) = \left[i n\cdot\partial + g_s n\cdot A_s(x_-) + g_s n\cdot A_c(x) \right] \xi(x)$$

$$\text{decouple} \rightarrow \left[i n\cdot\partial + g_s n\cdot A_s(x_-) + g_s S_n(x_-)\, n\cdot A_c^{(0)}(x)\, S_n^{+}(x_-) \right] S_n(x_-)\, \xi^{(0)}(x)$$

$$= \Big[S_n(x_-)\, i n\cdot\partial + \underbrace{\left(i n\cdot\partial\, S_n(x_-) + g_s n\cdot A_s(x_-) S_n(x_-) \right)}_{= i n\cdot D_{s-}\, S_n(x_-) = 0} + g_s S_n(x_-)\, n\cdot A_c^{(0)}(x) \Big] \xi^{(0)}(x)$$

$$= S_n(x_-) \left[i n\cdot\partial + g_s n\cdot A_c^{(0)}(x) \right] \xi^{(0)}(x)$$

$$= S_n(x_-)\, i n\cdot D_c^{(0)}\, \xi^{(0)}(x)$$

Then we have

$$\bar{\xi} \frac{\not{\bar{n}}}{2} i n\cdot D \xi = \bar{\xi}^{(0)} \frac{\not{\bar{n}}}{2} i n\cdot D_c\, \xi^{(0)}$$

The soft gluon field no longer appears in the collinear Lagrangian.

Key step towards factorization!

• Factorization of cross section at operator level

match the current $J^\mu = \bar\psi \gamma^\mu \psi$ onto an effective current operator in SCET containing a collinear quark and an anti-collinear anti-quark

$$J^\mu(x) \to C_V(-s-i0)\, \bar\chi_n(x)\, \gamma_\perp^\mu\, \chi_{\bar n}(x) \qquad \gamma_\perp^\mu = g_\perp^{\mu\nu}\gamma_\nu\,, \quad g_\perp^{\mu\nu} = g^{\mu\nu} - \frac{\bar n^\mu n^\nu + n^\mu \bar n^\nu}{2}$$

Gauge invariant $\underline{\chi_n(x)} = W_n^\dagger(x)\,\xi_n(x)$, with $\xi_n(x) = \frac{\not n \not{\bar n}}{4}\psi_n(x)$ and $W_n(x) = \mathbb{P}\exp\left[ig\int_{-\infty}^0 dt\, \bar n\cdot A(x+t\bar n)\right]$

decouple with soft interaction

$$\chi_n(x) \to S_n(x_-)\,\chi_n(x) \qquad\qquad \text{with} \quad S_n(x) = \mathbb{P}\, e^{ig_s\int_{-\infty}^0 dt\; n\cdot A_s^a(x+tn)T^a}$$
$$\chi_{\bar n}(x) \to S_{\bar n}(x_+)\,\chi_{\bar n}(x)$$

States : $|X\rangle \to |X_c\rangle|X_{\bar c}\rangle|X_s\rangle$

At the operator level, the cross section can be written as

$$\frac{d\sigma}{d\tau} = \frac{1}{2s}\frac{1}{s^2}L_{\mu\nu}e_q^2\sum_X\int d^4x\, e^{iq\cdot x}\langle 0|J^{\nu\dagger}(x)|X\rangle\langle X|J^\mu(0)|0\rangle\,\delta\left(\tau - \frac{n\cdot P_{X,R}+\bar n\cdot P_{X,L}}{s}\right)$$

$$= \frac{1}{2s}\frac{1}{s^2}e_q^2 L_{\mu\nu}|C_V(-s-i0)|^2\sum_X\int d^4x\, e^{iq\cdot x}\underline{\langle 0|\bar\chi_{\bar n}(x)\gamma_\perp^\nu\chi_n(x)|X\rangle\langle X|\bar\chi_n(0)\gamma_\perp^\mu\chi_{\bar n}(0)|0\rangle}\;\underline{\delta\left(\tau - \frac{n\cdot P_{X,R}+\bar n\cdot P_{X,L}}{Q}\right)}$$

$$\boxed{\begin{aligned} &= \langle 0|(\bar\chi_{\bar n} S_{\bar n}^\dagger \gamma_\perp^\nu S_n \chi_n)(x)\;|X\rangle\langle X|(\bar\chi_n S_n^\dagger\gamma_\perp^\mu S_{\bar n}\chi_{\bar n})(0)\,|0\rangle \\ &= (\gamma_\perp^\nu)_{\alpha\beta}(\gamma_\perp^\mu)_{\rho\sigma}\langle 0|T[S_{\bar n}^\dagger S_n(x)]|X_s\rangle\langle X_s|T[S_n^\dagger S_{\bar n}(0)]|0\rangle \\ &\quad\times\langle 0|\bar\chi_{\bar n,\alpha}(x)|X_{\bar c}\rangle\langle X_{\bar c}|\chi_{\bar n,\sigma}(0)|0\rangle\;\langle 0|\chi_{n,\beta}(x)|X_c\rangle\langle X_c|\bar\chi_{n,\rho}(0)|0\rangle\end{aligned}}$$

$$= \delta\left(\tau - \frac{P_{X_c}^2+P_{X_{\bar c}}^2+\sqrt{s}(\omega_++\omega_-)}{s}\right)$$

Jet : $\sum_{X_c}\langle 0|\chi_{n,\beta}(x)|X_c\rangle\langle X_c|\bar\chi_{n,\rho}(0)|0\rangle = \left(\frac{\not n}{2}\right)_{\beta\rho}\int\frac{d^4p_c}{(2\pi)^3}\,\theta(p_c^0)\,(\bar n\cdot p_c)\,J(p_c^2)\,e^{-ix\cdot p_c}$

Soft : $\sum_{X_s}\langle 0|\bar T[S_{\bar n}^\dagger S_n(x)]|X_s\rangle\langle X_s|T[S_n^\dagger S_{\bar n}(0)]|0\rangle = N_c\int_0^\infty d\omega_+\int_0^\infty d\omega_-\, e^{-i(\omega_+\bar n/2+\omega_- n/2)\cdot x}\; S_{\text{hemi.}}(\omega_+,\omega_-)$

$$\Rightarrow \frac{d\sigma}{d\tau} = \frac{1}{2s}\int_0^\infty d\omega_+\int_0^\infty d\omega_-\int\frac{d^4p_c}{(2\pi)^3}\int\frac{d^4p_{\bar c}}{(2\pi)^3}\;\frac{1}{s^2}e_q^2 N_c L_{\mu\nu}\mathrm{Tr}\left(\gamma_\perp^\nu\frac{\not n}{2}\gamma_\perp^\mu\frac{\not{\bar n}}{2}\right)(\bar n\cdot p_c)(n\cdot p_{\bar c})\;\int d^4x\, e^{i(q-p_c-p_{\bar c}-\omega_+\bar n/2-\omega_- n/2)\cdot x}$$

$$\times\; |C_V(-s-i0)|^2\, J(p_c^2)\,J(p_{\bar c}^2)\,S_{\text{hemi.}}(\omega_+,\omega_-)\;\delta\left(\tau - \frac{p_c^2+p_{\bar c}^2+\sqrt{s}(\omega_++\omega_-)}{s}\right)$$

$$= \underbrace{\int_0^\infty d\omega}_{\omega=\omega_++\omega_-}\int_0^\omega d\omega_-\int_0^\infty dp_L^2\int_0^\infty dp_R^2\;\overbrace{\boxed{\frac{1}{2s}\int\frac{d^4p_c}{(2\pi)^3}\delta(p_c^2-p_R^2)\int\frac{d^4p_{\bar c}}{(2\pi)^3}\delta(p_{\bar c}^2-p_L^2)\,|\mathcal{M}_0|^2\,(2\pi)^4\delta^{(4)}\left(q-p_c-p_{\bar c}-\omega_+\frac{\bar n}{2}-\omega_-\frac{n}{2}\right)}}^{=\sigma_0+\mathcal{O}(\tau)}$$

$$\times\; H(-s-i0)\,J(p_L^2)\,J(p_R^2)\,S_{\text{hemi.}}(\omega-\omega_-,\omega_-)\;\delta\left(\tau-\frac{p_L^2+p_R^2+\sqrt{s}\,\omega}{s}\right)$$

$$\Rightarrow \frac{d\sigma}{d\tau} = \sigma_0\, H(-s-i0)\int_0^\infty dp_L^2\, J(p_L^2)\int_0^\infty dp_R^2\, J(p_R^2)\int_0^\infty d\omega\;\underline{S_T(\omega)}\;\delta\left(\tau-\frac{p_L^2+p_R^2+\sqrt{s}\,\omega}{s}\right)$$

$$S_T(\omega) = \int_0^\omega d\omega_-\, S_{\text{hemi.}}(\omega-\omega_-,\omega_-)$$

4. Fixed-order Calculation of matrix elements

- Jet function at NLO

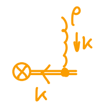
Collinear quark field $\xi(x) = \frac{\slashed{n}\slashed{\bar{n}}}{4}\psi(x)$

definition : $\frac{\slashed{n}}{2}\,\bar{n}\cdot p\, J_q(p^2) = \frac{1}{\pi}\,\mathrm{Im}\left[i\int d^dx\, e^{-ip\cdot x}\langle 0|\,T\left\{ W^\dagger(0)\,\boxed{\xi(0)}\,\boxed{\bar{\xi}(x)}\,W(x)\right\}|0\rangle \right]$

where $W(x)$ is collinear Wilson line

$$W(x) = \mathbb{P}\exp\left[ig\int_{-\infty}^{0} dt\; \bar{n}\cdot A(x+t\bar{n})\right]$$

Feynman rule:



$$= ig\int_{-\infty}^{0} dt\; \bar{n}^\rho \int\frac{d^dk}{(2\pi)^d}\, e^{ik(x+t\bar{n})}$$
$$= \int\frac{d^dk}{(2\pi)^d}\, e^{ik\cdot x}\; ig\,\bar{n}^\rho\, \frac{1}{i\bar{n}\cdot k}\, e^{it\bar{n}\cdot k}\Big|_{-\infty}^{0}$$
$$= \int\frac{d^dk}{(2\pi)^d}\, e^{ik\cdot x}\; ig\,\frac{i\bar{n}^\rho}{-\bar{n}\cdot k}$$

LO : $\frac{\slashed{n}}{2}\,\bar{n}\cdot p\, J_q^{(0)}(p^2) = \frac{1}{\pi}\,\mathrm{Im}\left[i\int d^dx\, e^{-ip\cdot x}\int\frac{d^dk}{(2\pi)^d}\, e^{ik\cdot x}\,\frac{\slashed{n}\slashed{\bar{n}}}{4}\,\frac{i\slashed{k}}{k^2+i0}\,\frac{\slashed{\bar{n}}\slashed{n}}{4}\right]$

$$= \frac{\slashed{n}}{2}(\bar{n}\cdot p)\cdot\frac{1}{\pi}\,\mathrm{Im}\left(-\frac{1}{p^2+i0}\right)$$
$$= \frac{\slashed{n}}{2}(\bar{n}\cdot p)\;\delta(p^2)$$

$$\frac{\slashed{n}\slashed{\bar{n}}}{4}\,\slashed{k}\,\frac{\slashed{\bar{n}}\slashed{n}}{4} = 2\,\bar{n}\cdot k\,\frac{\slashed{n}\slashed{\bar{n}}\slashed{n}}{16} = \frac{\slashed{n}}{2}(\bar{n}\cdot k)$$

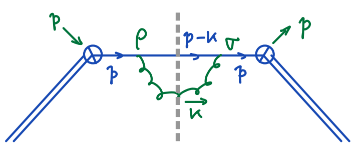
$$\delta(p^2) = \frac{1}{2\pi i}\left(\frac{1}{p^2-i0} - \frac{1}{p^2+i0}\right)$$

NLO:

Cutkosky rule :

1. Cut through the digram in all possible way such that the cut propagators can be put on shell
2. $\frac{1}{\pi}\,\mathrm{Im}[iA] \to \frac{1}{2\pi}\, A\left(\frac{1}{p^2+i0}\to(-2\pi i)\,\delta(p^2),\; \frac{1}{(p-k)^2+i0}\to(-2\pi i)\,\delta((p-k)^2)\right)$
3. sum the contributions of all possible cuts.

Diag. a



$$= \frac{1}{\pi}\,\mathrm{Im}\left[i\int\frac{d^dk}{(2\pi)^d}\,\frac{-ig_{\rho\sigma}}{k^2+i0}\,\frac{\slashed{n}\slashed{\bar{n}}}{4}\,\frac{i\slashed{p}}{p^2+i0}\, ig\gamma^\sigma\,\frac{i(\slashed{p}-\slashed{k})}{(p-k)^2+i0}\, ig\gamma^\rho\,\frac{i\slashed{p}}{p^2+i0}\,\frac{\slashed{\bar{n}}\slashed{n}}{4}\right] C_F$$

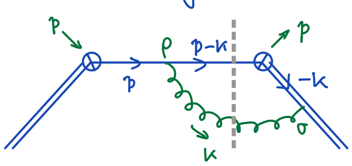
$$= \frac{1}{2\pi}\int\frac{d^dk}{(2\pi)^d}\,(2\pi)\,\delta(k^2)\,(2\pi)\,\delta((p-k)^2)\left[-g_{\rho\sigma}\,\frac{\slashed{n}\slashed{\bar{n}}}{4}\,\frac{i\slashed{p}}{p^2}\, ig\,\gamma^\sigma(\slashed{p}-\slashed{k})\, ig\gamma^\rho\,\frac{i\slashed{p}}{p^2}\,\frac{\slashed{\bar{n}}\slashed{n}}{4}\right] C_F$$

$$= \frac{e^{\varepsilon\gamma_E}}{(4\pi)^\varepsilon}\,\frac{1}{(2\pi)^{3-2\varepsilon}}\,\frac{2\pi^{1-\varepsilon}}{\Gamma(1-\varepsilon)}\quad \frac{1}{4}\int_0^\infty dk_+\int_0^\infty dk_-\,(k_+k_-)^{-\varepsilon}\,\delta(p^2-p_+k_--p_-k_+)\quad C_Fg^2\,2(1-\varepsilon)\,p^2\,k_-\,\frac{1}{(p^2)^2}\,\frac{\slashed{n}}{2}$$

$$= \frac{\slashed{n}}{2}\,\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\,\frac{C_F\alpha_s}{4\pi}\,\underline{\int_0^{p_-}dk_-\,(p_-)^{-1}\left[p_-^{-1}\left(p^2-\frac{p^2}{p^-}k_-\right)\right]^{-\varepsilon}k_-^{-\varepsilon}\;2(1-\varepsilon)\,\frac{k_-}{p^2}} = 2(1-\varepsilon)\,p_-\,(p^2)^{-1-\varepsilon}\int_0^1 dx\,(1-x)^{-\varepsilon}\,x^{1-\varepsilon}$$

$$\Rightarrow\; \frac{\slashed{n}}{2}(\bar{n}\cdot p)\, I_a = \frac{\slashed{n}}{2}(\bar{n}\cdot p)\,(p^2)^{-1-\varepsilon}\,\frac{C_F\alpha_s}{4\pi}\,\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\,\frac{2\Gamma^2(2-\varepsilon)}{\Gamma(3-2\varepsilon)}$$

Diag. b



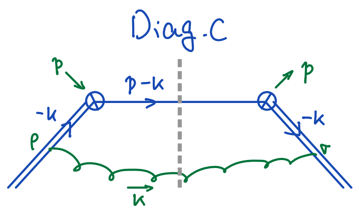
T^a acts on anti-quark in final state

$$= \frac{1}{\pi}\,\mathrm{Im}\left[i\int\frac{d^dk}{(2\pi)^d}\,\frac{-ig_{\rho\sigma}}{k^2+i0}\,\frac{\slashed{n}\slashed{\bar{n}}}{4}\,\frac{i(\slashed{p}-\slashed{k})}{(p-k)^2+i0}\, ig\gamma^\rho\,\frac{i\slashed{p}}{p^2+i0}\,\frac{\slashed{\bar{n}}\slashed{n}}{4}\, ig\,\frac{i\bar{n}^\sigma}{-\bar{n}\cdot k}\right]\underline{(-C_F)}$$

$$= \frac{1}{2\pi}\int\frac{d^dk}{(2\pi)^d}\,(2\pi)\,\delta(k^2)\,(2\pi)\,\delta((p-k)^2)\left[-g_{\rho\sigma}\,\frac{\slashed{n}\slashed{\bar{n}}}{4}\,(\slashed{p}-\slashed{k})\, ig\gamma^\rho\,\frac{i\slashed{p}}{p^2}\,\frac{\slashed{\bar{n}}\slashed{n}}{4}\, ig\,\frac{i\bar{n}^\sigma}{\bar{n}\cdot k}\right](-C_F)$$

$$= \frac{\slashed{n}}{2}\,\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\,\frac{C_F\alpha_s}{4\pi}\,\underline{\int_0^{p_-}dk_-\,(p_-)^{-1}\left[p_-^{-1}\left(p^2-\frac{p^2}{p^-}k_-\right)\right]^{-\varepsilon}k_-^{-\varepsilon}\;\frac{2p_-(p_--k_-)}{p^2\,k_-}} = 2\,(p^2)^{-1-\varepsilon}\int_0^1 dx\,(1-x)^{1-\varepsilon}\,x^{-1-\varepsilon}$$

$$\Rightarrow\; \frac{\slashed{n}}{2}(\bar{n}\cdot p)\, I_b = \frac{\slashed{n}}{2}(\bar{n}\cdot p)\,(p^2)^{-1-\varepsilon}\,\frac{C_F\alpha_s}{4\pi}\,\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\,\frac{2\Gamma(2-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(2-2\varepsilon)}$$



$$= \frac{1}{\pi} \mathrm{Im} \left[i \int \frac{d^d k}{(2\pi)^d} \frac{-i g_{\rho\sigma}}{k^2+i0} \frac{\not{n}\not{\bar{n}}}{4} \right.$$

$$= \frac{1}{2\pi} \int \frac{d^d k}{(2\pi)^d} (2\pi)\delta(k^2)(2\pi)\delta((p-k)^2) \left[-g_{\rho\sigma} \frac{\not{n}\not{\bar{n}}}{4} (\not{p}-\not{k}) \frac{\not{\bar{n}}\not{n}}{4}\, ig \frac{i\bar{n}^\sigma}{\bar{n}\cdot k}\, ig \frac{i\bar{n}^\rho}{\bar{n}\cdot k} \right]$$

$$\Rightarrow \quad I_c = 0$$

using expansion of star-function

$$\frac{1}{p^2}\left(\frac{p^2}{\mu^2}\right)^{-\varepsilon} = -\frac{1}{\varepsilon}\delta(p^2) + \left(\frac{1}{p^2}\right)_* - \varepsilon\left(\frac{1}{p^2}\ln\frac{p^2}{\mu^2}\right)_* + \mathcal{O}(\varepsilon^2)$$

with

$$\int_0^{Q^2} dp^2\, [f(p^2)]_*\, g(p^2) = \int_0^{Q^2} dp^2\, [g(p^2) - g(0)]\, f(p^2) + g(0)\int_{\mu^2}^{Q^2} dp^2\, f(p^2)$$

NLO jet function is given by

$$J^{(1)}(p^2) = I_a + 2I_b + I_c$$

$$= \frac{C_F\alpha_s}{4\pi}\left[-\frac{1}{\varepsilon}\delta(p^2) + \left(\frac{1}{p^2}\right)_* - \varepsilon\left(\frac{1}{p^2}\ln\frac{p^2}{\mu^2}\right)_* + \mathcal{O}(\varepsilon^2)\right] \frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\left(\frac{2\Gamma^2(2-\varepsilon)}{\Gamma(3-2\varepsilon)} + 2\cdot\frac{2\Gamma(2-\varepsilon)\Gamma(-\varepsilon)}{\Gamma(2-2\varepsilon)}\right)$$

$$\Rightarrow \quad J^{(1)}(p^2) = \frac{C_F\alpha_s}{4\pi}\left\{\frac{4}{\varepsilon^2}\delta(p^2) + \frac{1}{\varepsilon}\left[-4\left(\frac{1}{p^2}\right)_* + 3\,\delta(p^2)\right] + 4\left(\frac{1}{p^2}\ln\frac{p^2}{\mu^2}\right)_* - 3\left(\frac{1}{p^2}\right)_* + (7-\pi^2)\,\delta(p^2)\right\}$$

- Soft function at NLO

$$S_T(\omega) = \sum_{X_s} \langle 0|\bar{T}[S^\dagger_{\bar{n}} S_n(0)]|X_s\rangle\langle X_s|T[S^\dagger_n S_{\bar{n}}(0)]|0\rangle\, \delta(\omega - \bar{n}\cdot p_{X,L} - n\cdot p_{X,R})$$

The calculation is totally the same with eikonal approximation.

• Computation of higher loop integrals

Scale separation in SCET leads to trivil scale dependent low energy matrix elements.

The techniques to compute trivial scale dependent Feynman integrals

- Mellin-Barnes representations
- Sector decomposition

} not easy to obtain analytical results for three-loop integrals

- Dimensional recurrence relations + IBP reduction + HyperInt

Automatical & Analytical

• Dimensional Recurrence Relations (DRR) hep-th/9606018 O.V. Tarasov

$$F(\vec{a}, d) = \sum_i C_i F_i(\vec{a}_i, d+2)$$

This linear relation can be derived from alpha-parameters representation of Feynman integrals

Feynman parameters representation (see Peskin's QFT)

$$\frac{1}{D_1^{a_1} D_2^{a_2} \cdots D^{a_N}} = \frac{\Gamma(a_1+a_2+\cdots+a_N)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_N)} \int_0^1 dx_1 \cdots \int_0^1 dx_N \frac{x_1^{a_1-1} x_2^{a_2-1} \cdots x_N^{a_N-1} \delta(1-x_1-x_2-\cdots x_N)}{(x_1 D_1 + x_2 D_2 + \cdots + x_N D_N)^A} \quad \text{with } A = a_1 + \cdots + a_N$$

Alpha parameters representation

$$\frac{1}{D_1^{a_1} D_2^{a_2} \cdots D^{a_N}} = \frac{i^{-A}}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_N)} \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_N \; \alpha_1^{a_1-1} \alpha_2^{a_2-1} \cdots \alpha_N^{a_N-1} e^{i[\alpha_1 D_1 + \alpha_2 D_2 + \cdots + \alpha_N D_N]}$$

for $n=1$, $\quad i^{-1}\int_0^\infty d\alpha \, e^{i\alpha D} = \frac{i^{-1}}{iD} e^{i\alpha D}\Big|_0^\infty = \frac{1}{D}$

$$\frac{i^{-n+1}}{\Gamma(n-1)} \int_0^\infty d\alpha \, \alpha^{n-2} e^{i\alpha D} = \frac{i^{-n+1}}{\Gamma(n)} \left[\alpha^{n-1} e^{i\alpha D}\Big|_0^\infty - iD \int_0^\infty d\alpha \, \alpha^{n-1} e^{i\alpha D} \right] = D \frac{i^{-n}}{\Gamma(n)} \int_0^\infty d\alpha \, \alpha^{n-1} e^{i\alpha D}$$

$$\Rightarrow \frac{i^{-n}}{\Gamma(n)} \int_0^\infty d\alpha \, \alpha^{n-1} e^{i\alpha D} = \frac{1}{D^n}$$

inserting identity $1 = \int_0^\infty \frac{d\lambda}{\lambda} \delta(1 - \frac{1}{\lambda}\sum_{n=1}^N \alpha_n)$, and changing variables $\alpha_n \to \lambda x_n$,

the Alpha parameters can be transformed to Feynman parameters

$$= \Gamma(A) \cdot i^{A-1} \left(\sum_{n=1}^N x_n D_n\right)^{1-A} \cdot i \left(\sum_{n=1}^N x_n D_n\right)^{-1}$$

$$\frac{1}{D_1^{a_1} D_2^{a_2} \cdots D^{a_N}} = \frac{i^{-A}}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_N)} \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \; x_1^{a_1-1} x_2^{a_2-1} \cdots x_N^{a_N-1} \delta(1 - \sum_{n=1}^N x_n) \int_0^\infty d\lambda \, \lambda^{A-1} e^{\lambda i \sum_{n=1}^N x_n D_n}$$

$$= \frac{\Gamma(A)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_N)} \int_0^\infty dx_1 \cdots \int_0^\infty dx_N \; x_1^{a_1-1} x_2^{a_2-1} \cdots x_N^{a_N-1} \delta(1 - \sum_{n=1}^N x) \left(\sum_{n=1}^N x_n D_n\right)^{-1}$$

Equivalent to Feynman parameters

Dimensional Recurrence Relation can be derived by use alpha parameter representation, because d-dependence of Alpha parameter is easier.

$$\prod_{\ell=1}^{L}\int d^d k_\ell \frac{1}{D_1^{a_1} D_2^{a_2}\cdots D^{a_N}} = \frac{i^{-A}}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_N)}\int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_N \left(\prod_n^N \alpha^{a_n-1}\right)\prod_{\ell=1}^{L}\int d^d k_\ell \, e^{i\sum_{n=1}^{N}\alpha_n D_n}$$

Propagator $D_n = \Big(\sum_\ell w_{n,\ell} \underset{\text{loop}}{k_\ell} + \sum_m \eta_{n,m} \underset{\text{external}}{q_m}\Big)^2 - m_n^2 \qquad w_{n,\ell}\,,\ \eta_{n,m} = 0, \pm 1$

$$\sum_{n=1}^{N}\alpha_n D_n = \sum_{\ell,n} k_\ell M_{\ell n}(\{\alpha\}) k_n - 2\sum_\ell k_\ell Q_\ell + \tilde{J}(\{\alpha\},\{q\}) - \sum_{n=1}^{N}\alpha_n m_n^2 \,, \qquad M = M^T$$

- Shift momenta to remove linear terms in k^μ : $k^\mu \to k^\mu + M^{-1}Q^\mu$

$$\sum_{n=1}^{N}\alpha_n D_n = [k^T + Q^T(M^{-1})^T]\, M\,(k + M^{-1}Q) - 2Q^T k + \tilde{J}(\{\alpha\},\{q\}) - \sum_n \alpha_n m_n^2$$

$$= k^T M(\{\alpha\}) k + J(\{\alpha\},\{q\}) - \sum_n \alpha_n m_n^2\,, \qquad \text{with } J(\{\alpha\},\{q\}) = \tilde{J}(\{\alpha\},\{q\}) + Q^T M^{-1} Q$$

- Diagonalize $M(\{\alpha\})$

$$\sum_{n=1}^{N}\alpha_n D_n = [V(\alpha)k]^\dagger\, V(\alpha)M(\alpha)V^\dagger(\alpha)\, V(\alpha)k + J(\{\alpha\},\{q\}) - \sum_n \alpha_n m_n^2$$

$$= \bar{k}^T M_{diag}(\alpha)\bar{k} + J(\{\alpha\},\{q\}) - \sum_n \alpha_n m_n^2$$

$$= \sum_{\ell=1}^{L}\lambda_\ell \bar{k}_\ell^2 + J(\{\alpha\},\{q\}) - \sum_n \alpha_n m_n^2 \qquad \bar{k}_\ell^\mu = \lambda_\ell^{-\frac{1}{2}}\tilde{k}_\ell^\mu$$

$$= \sum_{\ell=1}^{L}\tilde{k}_\ell^2 + J(\{\alpha\},\{q\}) - \sum_n \alpha_n m_n^2$$

$$\Rightarrow \prod_{\ell=1}^{L}\int d^d k_\ell = \underbrace{(\det V)^d}_{=1}\prod_{\ell=1}^{L}\int d^d \bar{k}_\ell = \prod_{\ell=1}^{L}\lambda_\ell^{-\frac{d}{2}}\prod_{\ell=1}^{L}\int d^d\tilde{k}_\ell = \underbrace{[\det M(\alpha)]^{-\frac{d}{2}}}_{\text{polynomial of } \underbrace{\alpha_i\cdots\alpha_j}_{L}}\prod_{\ell=1}^{L}\int d^d\tilde{k}_\ell$$

$$\Rightarrow \prod_{\ell=1}^{L}\int d^d k_\ell \frac{1}{D_1^{a_1} D_2^{a_2}\cdots D^{a_N}} = \frac{i^{-A}}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_N)}\int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_N \left(\prod_n^N \alpha^{a_n-1}\right)[\det M(\alpha)]^{-\frac{d}{2}}$$

$$\times \prod_{\ell=1}^{L}\int d^d\tilde{k}_\ell \exp\Big[i\Big(\sum_{\ell=1}^{L}\tilde{k}_\ell^2 + J(\{\alpha\},\{q\}) - \sum_n \alpha_n m_n^2\Big)\Big]$$

$\int d^d k\, e^{ik^2} = i\pi^{\frac{d}{2}}(-i)^{\frac{d}{2}}$

k^μ is in Minkovski space

$$= \frac{i^{-A}}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_N)}\int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_N \left(\prod_n^N \alpha^{a_n-1}\right)[\det M(\alpha)]^{-\frac{d}{2}}$$

$$\times (i\pi^{\frac{d}{2}})^L (-i)^{\frac{Ld}{2}} \exp\Big[i\Big(J(\{\alpha\},\{q\}) - \sum_n \alpha_n m_n^2\Big)\Big]$$

define $F(\vec{a}\,;d) = \prod_{\ell=1}^{L}\int \frac{d^d k_\ell}{i\pi^{d/2}}\frac{1}{D_1^{a_1} D_2^{a_2}\cdots D^{a_N}}$

$$\Rightarrow F(\vec{a}\,;d) = \frac{i^{-A-\frac{Ld}{2}}}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_N)}\int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_N \left(\prod_n^N \alpha^{a_n-1}\right)[\det M(\alpha)]^{-\frac{d}{2}} \exp\Big[i\Big(J(\{\alpha\},\{q\}) - \sum_n \alpha_n m_n^2\Big)\Big]$$

Here we can see d-dependence only appears in $\underbrace{\det M(\alpha)}_{L\times L \text{ matrix}}$ and prefactor

So an integral in d-dimension can be rewritten as

$$F(\vec{a};d) = \frac{i^L \cdot i^{-A-\frac{L(d+2)}{2}}}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_N)} \int_0^\infty d\alpha_1 \cdots \int_0^\infty d\alpha_N \left(\prod_n^N \alpha^{a_n-1}\right)$$

$$\times \det M(\alpha) \left[\det M(\alpha)\right]^{-\frac{d+2}{2}} \exp\left[i\left(J(\{\alpha\},\{q\}) - \sum_n \alpha_n m_n^2\right)\right]$$

$$\Rightarrow F(\vec{a};d) = i^L \underbrace{\det M\left(\left\{i\frac{\partial}{\partial m_n^2}\right\}\right)}_{\text{polynomial of } \underbrace{\alpha_i\cdots\alpha_j}_{L}} F(\vec{a};d+2)$$

$$\Rightarrow F(\vec{a};d) = (-1)^L \det M\left(\left\{\frac{\partial}{\partial m_n^2}\right\}\right) F(\vec{a};d+2)$$

$\det M\left(\frac{\partial}{\partial m^2}\right)$ denotes $\det M(\alpha)$ with replacement $\alpha_i \to \partial/\partial m_i^2$

Actually, $\partial/\partial m_n^2$ can be rewritten as raising operator $\hat{n}^+$ of propagator D_i

$$\frac{\partial}{\partial m_i^2} F(\vec{a};d+2) = (-a_n)(-1) F(\{a_1,\ldots,a_n+1,\cdots,a_N\};d+2)$$

$$= a_n \cdot \hat{n}^+ F(\vec{a};d+2)$$

So mass term is not necessary in a propagator,

Finally, the dimensional recurrence relation is given by

$$F(\vec{a};d) = (-1)^L \det M(\{a_n \hat{n}^+\}) F(\vec{a};d+2)$$

Example:

$$I(a,b,c;d) = \left(i\pi^{\frac{d}{2}}\right)^{-1} \int d^d k \frac{1}{(k^2)^a[(k+p_1)^2]^b[(k-p_2)^2]^c} = (-1)^{a+b+c} \frac{\Gamma(a+b+c-\frac{d}{2})\Gamma(\frac{d}{2}-a-c)\Gamma(\frac{d}{2}-a-b)}{\Gamma(b)\Gamma(c)\Gamma(d-a-b-c)} (-s)^{\frac{d}{2}-a-b-c}$$

$$\det M = \det(\alpha_1+\alpha_2+\alpha_3) = \alpha_1+\alpha_2+\alpha_3$$

$$\Rightarrow I(a,b,c;d) = (-1)^1 \cdot (a\hat{a}_1^+ + b\hat{a}_2^+ + c\hat{a}_3^+) I(a,b,c;d+2)$$

$$= -a I(a+1,b,c;d+2) - b I(a,b+1,c;d+2) - c I(a,b,c+1;d+2)$$

Shift to higher dimension ($d = 6, 8, 10, \ldots$) helps to decrease infrared (IR) divergences

e.g. $\int d^d k \frac{1}{(k^2)^a} = i\Omega_d \int dk_E\, k_E^{d-1}(-k_E^2)^{-a} = i(-1)^a \Omega_d \int dk_E\, k_E^{d-2a-1}$, $d = n - 2\varepsilon$, $n = 2, 4, 6, 8, \cdots$

$d \leqslant 2a$, $\varepsilon < 0$ regularize IR poles arising from $k_E \to 0$

$d \geqslant 2a$, $\varepsilon > 0$ regularize UV poles arising from $k_E \to \infty$

$$\int d^dk \frac{(-1)^{a+b+c}}{(k^2)^a[(k+p_1)^2]^b[(k-p_2)^2]^c} = i\pi^{\frac{d}{2}} \frac{\Gamma(a+b+c-\frac{d}{2})}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^1 dy\, y^{\frac{d}{2}-a-c-1} (1-y)^{\frac{d}{2}-b-1} \int_0^1 dz\, z^{\frac{d}{2}-a-b-1} (1-z)^{a-1} (-s)^{\frac{d}{2}-a-b-c}$$

positive sign

exponents of propagators have negative signs

increase d $\Leftrightarrow$ decrease ε $\Rightarrow$ suppress IR

increase exponents $\{a, b, c\}$ $\Leftrightarrow$ increase ε $\Rightarrow$ suppress UV poles

- Calculation of $\int d^dk\, e^{ik^2}$, k^μ is in Minkovski space

$$\int d^dk\, e^{ik^2} = \int d\lambda\, \delta(1-\lambda) \int d^dk\, e^{i\lambda k^2} = \int d\lambda \frac{1}{2\pi} \int dx\, e^{ix(1-\lambda)} \int d^dk\, e^{i\lambda k^2} = \frac{1}{2\pi} \int dx\, e^{ix} \int d^dk \int_0^\infty d\lambda\, e^{i\lambda(k^2-x)}$$

$$= \frac{1}{2\pi} \int dx\, e^{ix} \int d^dk \frac{+i}{k^2-x+i0} = \frac{1}{2\pi} \int dx\, e^{ix} \int d^dk_E \frac{1}{k_E^2+x-i0} = \frac{1}{2\pi} \int dx\, e^{ix}\, i \int_0^\infty d\lambda \int d^dk_E\, e^{i\lambda(-k_E^2-x+i0)}$$

$$= i\int d^dk_E\, e^{i(-k_E^2+i0)} = i\,\Omega_d \frac{1}{2} \int_0^\infty dk_E^2\, (k_E^2)^{\frac{d}{2}-1} e^{-ik_E^2} = i\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{1}{2} (-i)^{\frac{d}{2}-1} \left[(\tfrac{d}{2}-1)(\tfrac{d}{2}-2)\cdots 1\right] \int_0^\infty dk_E^2\, e^{-ik_E^2} = i\frac{\pi^{\frac{d}{2}}}{\cancel{\Gamma(\frac{d}{2})}} (-i)^{\frac{d}{2}} \cancel{\Gamma(\tfrac{d}{2})}$$

$$\Rightarrow \int d^dk\, e^{ik^2} = i\pi^{\frac{d}{2}} (-i)^{\frac{d}{2}}$$

- Integration-by-part (IBP) reduction

An integral consists of L internal momenta $\{k\}$ and M external momenta $\{q\}$

$$F(\vec{a}) = \prod_{l=1}^{L} \int d^dk_l \frac{1}{\prod_n^N [D_n(\{k\},\{q\})]^{a_n}}$$

using IBP relation, we have $\displaystyle \prod_{l=1}^{L} \int d^dk_l \frac{\partial}{\partial k_j^\mu} \frac{p^\mu}{\prod_n^N [D_n(\{k\},\{q\})]^{a_n}} = 0$ $\qquad p^\mu = k_l^\mu$ or q_i^μ

There are $L(L+M)$ IBP relations in total. ie $\frac{\partial}{\partial k_j^\mu} k_i^\mu$ and $\frac{\partial}{\partial k_j^\mu} q_i^\mu$

$$\prod_{l=1}^{L} \int d^dk_l \frac{\partial}{\partial k_j^\mu} \left[p^\mu \frac{1}{\prod_n^N [D_n(\{k\},\{q\})]^{a_n}} \right] = 0 \quad \Rightarrow \quad -\sum a_n \hat{O}_n^+ \hat{O}_m^- F(\vec{a}) = 0$$

Solve linear equation system, one can express $F(\vec{a})$ by linear combination of master integrals (MIs)

- Stratergy to compute Feynman integrals by using method of DRR

1. Relate MIs $\vec{G}(d)$ to the integrals $\{I(d+2)\}$ with DRR method

   Mathematica package LiteRed. by R.N. Lee

2. Reduce $\{I(d+2)\}$ to MIs $\vec{G}(d+2)$ with IBP method

   FIRE6 by Smirnov ; Kira2.0 by J. Usovitsch

3. Build linear relations between MIs in d and $d+2$,

   i.e. $\vec{G}(d+2) = \mathbb{A}(d)\cdot\vec{G}(d)$, ← DRR relation

4. Find finite integrals $F_i(d_i)$ $(d_i = 4, 6, 8, \cdots)$ by shifting dimension and increasing exponents of propagators

   Reduze2. by von Manteuffel

5. Using IBP reduction to get linear relation

$$F_i(d_i) = \sum C_{ij}\, G_j(d_i)$$

6. Using DRR relation, we can get

$$\vec{G}(d) = \mathbb{M}(\varepsilon)\,\vec{F}(\{d\})$$

7. Evaluate finite basis $F_i(d_i)$ HyperInt by E. Panzer

Example: Two-loop jet function